AN INTRODUCTION TO MATHEMATICAL THOUGHT

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by

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PREFACE

The present textbook is an expansion and revision of an experimental text with the same title, lithoprinted in 1948. The latter, in turn, was an outgrowth of mimeographed material prepared by the writer for use in an elementary course at Hofstra College, entitled *Foundations of Mathematical Thought.*

The chief aim of the text is to provide a unified and substantial approach to the logical structure of mathematics, and to develop a corresponding philosophical point of view toward mathematical knowledge. This is carried out by emphasis both on postulational foundations and on the process of logical reasoning itself, together with applications to science and other fields of thought.

However, the text is not intended to supply a technical treatment either of logic or of the foundations of mathematics. The presentation is introductory and somewhat exploratory in nature.

The text is organized in two parts. Part I is devoted primarily to philosophical aspects, historical background, elementary symbolic logic, and connections with scientific method.

Part II is largely concerned with modern postulational (or "axiomatic") methods, abstract postulational systems (including a finite geometry and various important algebraic systems), foundations of standard branches of mathematics, and some modern foundational points of view.

As preparation for the text it seems essential that students should have had at least plane geometry and intermediate algebra in high school. Trigonometry and college or advanced high school algebra would be desirable.

It is believed that by a suitable choice of chapters, or sections within chapters, the text can be adapted to various types of courses with different levels of difficulty — such as general education courses, background courses for prospective teachers of secondary mathematics, and introductory foundational courses for specialists in pure mathematics.

Three chapters (Chapters 4, 9, and 10) have been added since the lithoprinted edition, and are definitely more advanced than the others. Chapter 4 includes propositional functions and leads into the logic of classes; Chapters 9 and 10 include groups, rings, Boolean algebras,

relations, and lattices. These chapters are marked by asterisks to indicate that they may be omitted without damage to the continuity; they are hardly intended for elementary courses.

Sample outlines for varied courses, based on the text, are suggested on page xviii.

There are many persons who have contributed, knowingly or unknowingly, to the completion of this book. I shall mention a few of them here.

First, I wish to thank Professor L. F. Ollmann, Chairman of the Department of Mathematics at Hofstra College, for assigning to me the elementary course which provided the stimulus for this text, and for allowing me complete freedom to develop the course in the classroom as I saw fit. Moreover, through his cooperation, it was possible for me to undertake the revision of the lithoprinted text during the academic year 1951–52. I am grateful also to those past students of mine who served as subjects of experimentation both before and after the lithoprinted edition was available.

For orientation toward the ideas presented in the text I owe much to two of my former teachers at Harvard University, Professors E. V. Huntington and Ralph Beatley, and to such authors as C. J. Keyser, John Wesley Young, and E. T. Bell. I believe that I should note the less tangible but real assistance I have received from exposure to the standards of mathematical exposition upheld by the late Professors W. F. Osgood and W. C. Graustein. Also, for stimulating an early interest in mathematics I cite the influence of my father, Edward Lincoln Stabler, to whom this work is dedicated.

For reading the lithoprinted edition critically and offering helpful suggestions for its revision I am greatly indebted to the following persons: Professor D. R. Davis of New Jersey State Teachers College at Montclair; Mr. Frank Hawthorne of Hofstra College; Professors R. L. Wilder and M. O. Reade of the University of Michigan. Professor Reade also commented on the revised and expanded manuscript. For special advice concerning Chapters 3 and 4, I am grateful to my colleague Mr. W. Lysle Marshall, now of the University of Delaware. Naturally, I remain responsible for any errors or deficiencies in the final text.

To the Editorial Department of the Addison-Wesley Press, particularly Mr. Warren Blaisdell and Mrs. Olga Crawford, I extend a special word of thanks. Furthermore, I appreciate the assistance

of the following: Miss Rita Vanderhoof for typing the final manuscript; Mr. Richard Lamm and Mr. Edward Pinkiert for suggestions from the student point of view; Mrs. Gladys Meyer Williams for aid in typing one of the earliest drafts of the manuscript; Miss Wilma Cope and Miss Mary Pawelko for help with proofreading; the Hofstra Mimeographing Department for mimeographing prior to the experimental edition; Edwards Brothers of Ann Arbor, Michigan, for lithoprinting the experimental edition.

For their kind permission to quote from copyrighted material I wish to thank the following: Professors Albert Einstein and Leopold Infeld for use of passages from their book, *The Evolution of Physics*, published by Simon and Schuster; The Macmillan Company for use of two problems taken from *A Primer of Formal Logic* by J. C. Cooley; Scripta Mathematica for use of a passage from *Thinking about Thinking* by C. J. Keyser; and the Yale University Press for use of a passage by Henry Margenau from *The Development of the Sciences*, edited by L. L. Woodruff.

Finally, I am obligated to my wife for invaluable aid at all stages of the text. She prepared the photographic copy for the lithoprinted edition and has served continually as typist, critic, proofreader and adviser.

E. R. S.

Hempstead, New York
October 1952

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INTRODUCTION

The purpose of this text is not to give training in standard mathematical techniques or to develop proficiency in the use of mathematics as a tool for the solution of applied problems, but rather to present introductory material for the dual study of the role of thought in mathematics and the role of mathematics in thought. Such a study should be significant for both mathematical specialists and non-specialists, for both prospective scientists and philosophers.

More specifically the aims of the text are: *first*, to stimulate the reader to expand his own philosophical point of view toward mathematics, especially its foundations and applications and to a certain extent its historical development; *second*, to consider the place of logical reasoning and the significance of logical structure in both mathematical thought and scientific thought; *third*, to investigate modern postulational concepts and methods, and some significant postulational systems in modern mathematics, in contrast with classical postulational organization as represented by Euclid; *fourth*, to suggest some applications of the topics studied to other fields of intellectual activity; *fifth*, to give some introduction to more advanced topics in the foundations of mathematics with the hope of encouraging further study and reading in this direction.

For convenience in grouping topics, the text is divided into two parts, of which the second is somewhat more advanced mathematically.

Part I, *Mathematical Thought in Relation to Logic and Science*, contains the following chapters devoted to the indicated topics.

Chapter 1 is a general introduction to the nature of mathematical knowledge and truth, based on illustrations including non-Euclidean geometries and novel kinds of arithmetic, and in particular showing the dependence of mathematical facts on agreements in the form of postulates, definitions, and methods of reasoning.

Chapter 2 discusses the approximate historical origin of logical systems or postulationally organized structures of thought in Euclid's *Elements*, and after evaluating Euclid by modern standards of rigor, considers the use of postulational organization in arithmetic and algebra, and its value in general.

Chapters 3 and 4 attempt an elementary analysis of the active

aspect of logical systems, that is, logical reasoning, from the standpoint of modern symbolic logic, and present examples of how not to reason, based on common logical fallacies.

Chapter 5 considers the application of logical reasoning, and, more generally, "reflective thinking," to scientific method and the formulation of scientific theories, with historical illustrations based on Newton and Einstein; it makes a comparison of scientific truth and mathematical truth.

Part II, *Some Significant Postulational Systems, Concepts, and Methods,* includes the following chapters, with the specified subject matter.

Chapter 6 develops an important part of elementary algebra as a logical system, postulationally organized after the manner of high school demonstrative geometry.

Chapter 7 illustrates the modern trend in postulational organization by presenting an abstract geometric system, in the form of a finite or miniature geometry.

Chapter 8 builds on the preceding two chapters to give a more detailed discussion of modern postulational concepts and methods.

Chapter 9 is devoted to a discussion of some postulational systems of wide bearing in mathematics and logic — namely, groups, rings, and Boolean algebras.

Chapter 10 is concerned with relations, order systems, and lattices, and their connections with previous topics.

Chapter 11 presents a survey of mathematical foundations, covering geometry, algebra, natural numbers, and logic, together with some modern points of view in the foundations of mathematics.

The learning methods of the text are dictated by the program outlined. Exercises are included which put emphasis on illustration of new ideas, analysis of reasoning processes, formulation of proofs, and development of points of view. Some of the exercises are essentially discussion questions; others are more nearly reading assignments. However, for simplicity, they are all designated merely as "exercises."

The following books are recommended for supplementary reading and reference:

M. Richardson, *Fundamentals of Mathematics,* Macmillan, 1941.
Cooley, Gans, Kline, and Wahlert, *Introduction to Mathematics,* Houghton Mifflin, Second Edition, 1949 (first edition, 1937).

They will be referred to respectively as "Richardson" and "Cooley, Gans." Both of these books have an aim similar to ours but their approach is different. Both books also contain surveys of certain topics found in standard textbooks on college algebra, analytic geometry, and calculus. The present text, however, concentrates on subject matter which is useful for our purpose and which is less readily available in a unified treatment elsewhere. Other valuable books, in addition to the two mentioned, are referred to as the occasion arises.

A bibliography is included which lists all books and articles referred to in the text and some selected additional references.

SUGGESTED OUTLINES

The following lists are purely suggestive of the possibilities for classwork in different types of courses. The designations "brief" and "normal" refer to courses meeting two and three hours a week respectively. It is assumed that in most courses considerable time will be desired for class discussion. (In some cases additional chapters or sections might be assigned entirely for outside reading.)

One-semester general education course (brief): Chapters 1, 2, 3, 5

One-semester course for prospective teachers of secondary mathematics (normal): Chapters 1, 2, 3, 5, 6, 7, part of 8

One-semester course for specialists in pure mathematics (normal): Chapters 1, 2, 3, 6, 7, 8, part of 11

Two-semester general education course (brief): Chapters 1, 2, 3, 5, 6, 7, part of 8, part of 11

Two-semester course for specialists in pure mathematics (normal): All chapters

PART I

MATHEMATICAL THOUGHT IN RELATION TO LOGIC AND SCIENCE

CHAPTER 1

THE NATURE OF MATHEMATICAL KNOWLEDGE

1–1 Introduction. The absolute truth viewpoint. We begin our discussion with consideration of a popular view concerning the nature of mathematical facts. It is commonly maintained that mathematics is the only science in which the results are absolutely true and unchangeable, and in which all the answers can be completely determined. Thus, if a person wishes to emphasize that he knows a given fact to be definitely the case, he may say that he is just as sure of the fact in question as he is that $2 \times 2 = 4$. This simple statement epitomizes what is considered to be the unqualified truth and certainty of all mathematical facts. In this chapter we shall investigate the degree of justification for this absolute truth point of view, with the aid of illustrations drawn from various mathematical sources.

On close reflection, just what is meant by claiming that it is an absolute fact that $2 \times 2 = 4$? Does this statement express something inherent in the nature of the real world, so fundamental that it is incapable of logical proof? Or is it a fact which can be proved logically by a method similar to that used in proving theorems of geometry? In the first case, the statement is actually taken as an unproved assumption, regardless of how fundamental it is held to be. In the second case, it must be recognized that a logical proof of any kind always falls back on previously accepted statements, which themselves either are proved from other statements, or else are assumed to be correct. Some of the statements in question will necessarily involve definitions or agreements as to the meaning of the symbols used. Whichever point of view we adopt toward $2 \times 2 = 4$ — whether we think of it as a fundamental principle or as a fact capable of logical proof — we see that it is either an assumption itself, or else that it is obtained by the use of logical reasoning from other assumptions and agreements.

Exercises

1. Attempt to formulate a proof that $2 \times 2 = 4$. Then compare your proof with the somewhat informal proof proposed below.

Proposed proof: We presuppose knowledge of the natural numbers (positive integers) used in counting, 1, 2, 3, 4, ... In this list of integers each integer is to be defined as equal to its predecessor "plus" 1. Thus $2 = 1 + 1$, $3 = 2 + 1$, $4 = 3 + 1$, etc. It is possible to give a general definition for adding any two natural numbers. Also, it is possible to arrive at a general principle of combining or grouping in adding three natural numbers a, b, c, to the effect that

$$(a + b) + c = a + (b + c).$$

Next, it is possible to define multiplication with natural numbers in such a way as to fall back on addition. In particular, we agree that

$$a \cdot 1 = a,\ a(b + 1) = ab + a.$$

Thus to obtain $2 \cdot 2$ we start with $2 \cdot 1 = 2$. Then by definition of 2 as $1 + 1$, we have

$$\begin{aligned} 2 \cdot 2 &= 2(1 + 1) \\ &= 2(1) + 2 \\ &= 2 + 2. \end{aligned}$$

Our problem is thus reduced to adding 2 and 2. We now use the definition of 2 again, together with the grouping principle for the three numbers a, b, c mentioned above, obtaining

$$\begin{aligned} 2 + 2 &= 2 + (1 + 1) \\ &= (2 + 1) + 1 \\ &= 3 + 1 = 4 \text{ by definition of the numbers 3 and 4.} \end{aligned}$$

This completes our proof.

2. In the proof given above that $2 \cdot 2 = 4$, identify the steps which made use of definitions and those which made use of assumptions (or principles not specifically proved here).

3. Could you conceive of any situation where it would be correct to write $2 \times 2 = 0$ or $2 \times 2 = 11$? (More light will be thrown on this later in the chapter.)

4. Is it true that 2 and 2, and 0 and 0, are the only pairs of numbers (equal or unequal) which give the same result when added as when multiplied? (Let x and y designate such a pair of numbers. Then we want $x + y = xy$. Solve for y in terms of x. Is there any value which cannot be assigned to x?)

5. Is it possible to have two unequal numbers which are the squares of each other? (Here we allow use of complex numbers. Let x and y be the required numbers. Then we want both $y = x^2$ and $x = y^2$. Thus $x = x^4$, $x^4 - x = 0$, $x(x^3 - 1) = 0$, etc.)

1–2 An addition table for days of the week. As an illustration of the need for revision of the popular attitude toward arithmetic, let us consider a new type of addition table in which the results will look somewhat different from usual.

This new table will be based on days of the week. It will show what day of the week we arrive at by counting forward a certain number of days from a given day of the week. First we number the days of the week in the obvious way, beginning with 1 for Sunday and ending with 7 for Saturday. If we wish to know which day of the week is 5 days after Wednesday (day number 4), we think

$$4 + 5 = 9,$$

and since

$$9 = 7 + 2,$$

we see that the day obtained is two days after Saturday, that is, it is day number 2, or Monday. The result may be summarized by writing

$$4 + 5 = 2 \text{ (instead of 9)}.$$

To avoid offending our sense of propriety we may think of the 4, 5, 2, and + in this statement as mere symbols having the meaning already indicated. Similarly, to show that 4 days after Friday is Tuesday we can write

$$6 + 4 = 3 \text{ (instead of 10)}.$$

The essential point is that we are not interested in numerical results greater than 7. Hence, we always cut down our answer by 7 (or a multiple of 7) until it is less than or at most equal to 7. Thus we would write

$$7 + 7 = 7.$$

If we want to count forward 24 days from Thursday we think of 24 days as $(21 + 3)$ days and write

$$5 + 3 = 1,$$

neglecting the 21 days (or three full weeks), giving Sunday as the result. Of course we might have written

$$5 + 24 = 1,$$

cutting down our answer by 28, but it seems simpler to neglect in advance the largest multiple of 7 contained in the number of days being added.

In view of these considerations, our addition table for days of the week can be restricted to the use of integers from 1 to 7 inclusive. The table is to be read from left to right as shown by the underscored entries corresponding to the sum

$$5 + 4 = 2.$$

We look for 5 in the column to the left of the vertical line and 4 in the row above the horizontal line. Opposite the 5 and below the 4 we find the "sum" 2.

+	1	2	3	4	5	6	7
1	2	3	4	5	6	7	1
2	3	4	5	6	7	1	2
3	4	5	6	7	1	2	3
4	5	6	7	1	2	3	4
5	6	7	1	2	3	4	5
6	7	1	2	3	4	5	6
7	1	2	3	4	5	6	7

Let us comment on certain contrasts between this table and the tables of ordinary arithmetic, aside from the obvious differences in some of the addition combinations.

In the first place, we use in the main body of the table no symbols except those contained in the vertical and horizontal margins. In technical language we say that the table is a *closed system*, meaning that every sum indicated by the table is one of the original symbols 1, 2, . . . , 7 contained in the border of the table. Other symbols such as 8, 9, 10, . . . , 14 are outsiders, so to speak. The system is "closed" in the sense that no outsiders are permitted to enter. This very simple concept of a closed system is an extremely important one, both in the foundations of mathematics and in numerous branches of mathematics.

In the second place, the symbol "7" has the property that when added to any of the other symbols the result is that other symbol. Thus, if a is any symbol chosen from the given list of symbols (1, . . . , 7), then we have $a + 7 = a$ (or $7 + a = a$). In ordinary arithmetic the number which has the same property as our symbol "7" is the number zero (0). From this standpoint, "7" can be designated as the zero-symbol or zero-element of the system.

In the third place, for each symbol in the table there is a unique corresponding symbol which, when added to the first, yields the zero-symbol. Although no negative signs are used, we notice that this property is similar to the property connecting any number with its negative in ordinary computation with positive and negative numbers. Let us call symbols in the table "opposite" if when added together their sum is the zero-symbol, 7. To "subtract" one symbol in the table from another, it will appear (from Exercises 2

and 3 below) that all we have to do is add the opposite symbol — just as in ordinary algebra we subtract a number by adding its negative.

Exercises

1. Use the table above to find the day of the week which is (a) 5 days after Wednesday, (b) 16 days after Saturday, (c) 87 days after Tuesday, (d) 173 days after Thursday.

2. Use the table to solve the equations (a) $x + 5 = 3$, (b) $4 + x = 1$, and interpret as problems involving days of the week.

3. If a and b stand for arbitrary symbols in the table, let us define $a - b$ to mean the solution of the equation $x + b = a$. Let $\bar{b}$ = "opposite" of b. Explain why $a - b = a + \bar{b}$.

4. Construct a table for addition of clock hours, using symbols from 1 to 12.

5. Use the table of Exercise 4 to answer the following questions: (a) What time of day will it be 41 hours after 11 A.M.? (b) What time of day will it be 79 hours after 8 P.M.? (c) What is the "zero-element" of the system? (d) What is the "opposite" of 9 in this system? Of 5? Of 12?

6. Suppose the calendar were so constructed that every month had 30 days. Then a uniform system could be used for adding days of the month. Fill out the following table to cover the situation in this case:

+	5	10	15	20	25	30
5						
10						
15						
20						
25						
30						

7. In trigonometry, angles are equivalent (not strictly equal), as far as their trigonometric functions are concerned, when they differ by a multiple of 360° or 2π radians. Construct a table for adding the following angles (in radians) in such a way that all the sums are recorded as *less than* 2π: $0, \pi/2, \pi, 3\pi/2$.

1–3 Modulo arithmetic. The closed tables of the preceding section illustrate what may be referred to as "modulo arithmetic." The table for addition of days of the week, and the table for addition of clock hours mentioned in Exercise 4, are essentially tables for addition "modulo 7" and "modulo 12," respectively. When we add modulo 7 we record only remainders above a multiple of 7, so that a modulo 7 addition table would be obtained from our table for days

of the week by changing every 7 to 0. This would be appropriate from another point of view also, as we have seen that the symbol 7 acts in the table like the ordinary number 0.

The simplest possible example of modulo arithmetic is the system of odd and even integers. Remembering that an even integer is one which is a multiple of 2 and an odd integer is one which is not, it is easy to fill out the following tables showing the results of adding and multiplying with odd and even integers:

+	even	odd
even	even	odd
odd	odd	even

×	even	odd
even	even	even
odd	even	odd

We now replace "even" by "0" and "odd" by "1" to designate the respective remainders of even and odd integers above a multiple of 2. We obtain a modulo 2 system indicated by the following tables. The *modulus* of the system is 2. (Note that the word *modulus* is a noun, while *modulo* is used as if it were an adjective.)

Modulo 2 tables

+	0	1
0	0	1
1	1	0

×	0	1
0	0	0
1	0	1

As further illustrations let us construct tables for addition and multiplication modulo 3 and modulo 4. In accordance with the above statements, the tables are constructed by recording only remainders above multiples of 3 or 4, as the case may be.

Modulo 3 tables

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Modulo 4 tables

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

×	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

In each of these systems we note that the symbol 0 behaves most of the time like the number 0 in ordinary arithmetic. In each case we have $a + 0 = 0 + a = a$. Also, always $0 \cdot a = a \cdot 0 = 0$. In the modulo 4 system, however, we have 0 occurring in an unusual way as a product, namely $2 \times 2 = 0$. The meaning of this statement is obvious enough: it simply says that 2 times 2 gives a result that is 0 more than a multiple of 4. Nevertheless, many persons, seeing this statement without the accompanying explanation, would be inclined to brand it as absurd, if not malicious! (Compare with Exercise 3 of Section 1–1.)

In ordinary arithmetic or algebra, if we have the product of two numbers equal to 0 we know that at least one of these numbers must be 0. But in modulo arithmetic we see that it is sometimes possible to have two *nonzero* symbols whose product is the zero symbol. This is the case for the pair of equal symbols, 2 and 2 in the modulo 4 system, or for 5 and 3 in the modulo 15 system. It is convenient to have a special name for symbols of this kind, as indicated by the following definition: two nonzero symbols (equal or unequal) whose product is the zero symbol are called *divisors of zero.*

Modulo arithmetic is helpful not only in broadening mathematical perspective, but also in the study of modern abstract algebra (which turns out to be quite different in point of view from traditional algebra). In a somewhat modified form, modulo arithmetic is an important part of that branch of higher mathematics dealing with integers and known as *theory of numbers.**

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* In theory of numbers the emphasis is on the concepts of *congruence* and *residue classes*, rather than on our modulo tables. For example, the numbers 13 and 25 are said to be "congruent" modulo 6, meaning that their difference is a multiple of 6, or (what amounts to the same thing) that each leaves the same remainder on division by 6. The notation used is $13 \equiv 25 \mod 6$ ("13 is congruent to 25 modulo 6"). Similarly, $13 \equiv 1 \mod 6$ and $25 \equiv 1 \mod 6$. All integers congruent to each other modulo 6 are said to belong to the same "residue class" modulo 6. Every integer will be congruent to one of the integers 0, 1, 2, 3, 4, 5 modulo 6. When we make modulo 6 addition and multiplication tables we are in a sense representing all integers, or residue classes modulo 6, by the restricted list of symbols 0, 1, . . . , 5. Thus when we say $2 \cdot 5 = 4$ in the modulo 6 multiplication table, we can interpret this to mean that any integer congruent to 2 multiplied by any integer congruent to 5 yields an integer congruent to 4, all modulo 6. This can easily be checked by elementary algebra. For further details consult any textbook on theory of numbers.

In Part II of this text we shall investigate the logical background of modulo arithmetic and divisors of zero in a more fundamental way than is possible at the present time. Here the important thing is to note that there are valid and useful numerical statements, apparently differing from those of ordinary arithmetic, yet arrived at by changing the meaning of certain symbols and operations in a restrained manner. Thus the justification of the absolute truth point of view seems considerably weakened.

EXERCISES

1. Construct tables of addition and multiplication for the modulo 5 system. Are there divisors of zero in this system?
2. Repeat Exercise 1 for the modulo 6 system.
3. Do the same for the modulo 7 system.
4. Give an explanation of why some modulo systems have divisors of zero and some do not.
5. In ordinary algebra if we solve the equation $x^2 = x$, we obtain $x^2 - x = 0$, then $x(x - 1) = 0$, whence $x = 0$ *or* $x = 1$. Can we arrive at the same conclusion (a) in modulo 5 arithmetic, (b) in modulo 6 arithmetic? Why or why not?

1–4 Change of number base. One aspect of ordinary arithmetic which is related somewhat to modulo arithmetic is the concept of a number base. Our method of symbolizing numbers is known as the system of Hindu-Arabic numerals (sometimes just Arabic numerals). The essential feature of the system is that of positional notation, in which each digit of a number indicates a coefficient of a corresponding power of the *number base*, ten. Hence the number 234 means

$$2(10^2) + 3(10^1) + 4,$$

or

$$2(10^2) + 3(10^1) + 4(10^0),$$

since by definition $10^0 = 1$. Thus we count up to the given number by tens and powers of ten. (Of course, if we were using modulo 10 arithmetic we would think of 234 as being reduced to 4, since it is 4 more than a multiple of 10.)

Traditionally we count by tens probably because of the number of fingers on two hands. But if a human being normally had *four* fingers, or perhaps *six*, on each hand we might equally reasonably count by the numbers which we call *eight* and *twelve*, respectively.

There is historical evidence that certain civilizations made use of number bases of *twenty* and *sixty*. Evidence of the influence of

twelve or sixty as a base in certain common units of measure should be easy to cite. Number words in our own and other languages give indication of mixed ideas as to desirable bases in the past. (A textbook on the history of mathematics can be referred to for more detailed information on these points.)

Regardless of history, we know that if we desire to do so we can symbolize numbers by use of the base *two*, *five*, *seven*, *twelve*, or any desired positive integer (other than 1). It is good training in open-mindedness in arithmetic to practice working with numbers using some of these other bases. It is important to note, however, that the numbers and operations remain the same as in ordinary arithmetic. The only change is in the appearance of numbers when different bases are used. The *method* of positional notation remains the same as in the Hindu-Arabic system.

For example, using base five, the number seventeen would be written as "32," since

$$\text{Seventeen} = 3(5^1) + 2.$$

Note that the only digits used in writing numbers with base five are 1, 2, 3, 4, 0. As in the ordinary base ten system, we have no single digit for the number base. Counting in the base five system would be as follows:

1, 2, 3, 4, 10, 11, 12, 13, 14, 20, 21, 22, 23, 24, 30, 31, 32, 33, 34, 40, 41, 42, 43, 44, 100, etc.

As a further illustration, let us change the ordinary number 478 to base five. First, we must take the highest power of 5 which is less than 478, namely, 125 or 5^3. The number of 125's contained in 478 is 3, with a remainder of 103. Thus

$$478 = 3(5^3) + 103.$$

Next we ask how many times 25 or 5^2 is contained in 103. This is 4 with a remainder of 3.

$$103 = 4(5^2) + 3.$$

Finally, 5^1 is contained 0 times in 3 with a remainder of 3, or $3(5^0)$, so combining steps we have

$$478 = 3(5^3) + 4(5^2) + 0(5^1) + 3(5^0).$$

In other words, 478 (base ten) = 3403 (base five)
or, for short,

$$478_{\text{ten}} = 3403_{\text{five}},$$

indicating the base in word form below the given number. As a check, simply evaluate again the number 3403_{five} by powers of 5 in ordinary arithmetic (as in the second equality above), obtaining, in ordinary notation,

$$3(125) + 4(25) + 0(5) + 3 = 478_{\text{ten}}.$$

EXERCISES

1. Express the following *ordinary* numbers using base *five:* 37, 89, 143, 2398.

2. Change the following numbers, with indicated number base, back to a number with base *ten:* 27_{eight}, $1{,}001{,}010_{\text{two}}$, 2222_{three}, 546023_{seven}, 8012_{nine}.

3. Express the following ordinary numbers using the indicated base: 43 — base *two*, 398 — base *seven*, 161 — base *three*, 1513 — base *six*.

4. Some persons believe that the number twelve would form a more desirable number base than the number ten. With twelve as a base, two additional symbols are needed, say "t" for ten and "e" for eleven.

(a) Change to base *ten* the following numbers written with base *twelve:* 2t5e, 309t.

(b) Change to base *twelve* the following ordinary numbers: 1136, 7045.

5. Make up multiplication tables for numbers written with the following bases (in each case going up to "10" times "10"): (a) base *five*, (b) base *six*, (c) base *seven*, (d) base *eight*.

6. (a) *Illustration.* Do the following multiplication problem using base *seven*, then check by doing the same problem with base *ten:*

$$54_{\text{seven}} \times 32_{\text{seven}} = 39_{\text{ten}} \times 23_{\text{ten}}$$

Base seven	*Base ten*
54	39
32	23
141	117
225	78
2421	897

$$\begin{aligned} \textit{Check: } 2421_{\text{seven}} &= 2(7^3) + 4(7^2) + 2(7^1) + 1 \\ &= 686 + 196 + 14 + 1 \\ &= 897_{\text{ten}}. \end{aligned}$$

In the same way do the following problems and check:

(b) $32_{\text{five}} \times 41_{\text{five}}$.

(c) $142_{\text{six}} \times 53_{\text{six}}$.

(d) 3e × 4t (base *twelve*).

(e) $17_{\text{eight}} \times 23_{\text{eight}}$.

7. Investigate the "Russian peasant method" of multiplication, which depends on the indirect use of a binary number base, i.e., base *two*. (See H. T. Davis, *College Algebra,* Prentice-Hall, 1940, Chapter XIX.)

1–5 Euclidean and non-Euclidean geometries. So far we have found need for modification of the absolute truth point of view, relying on illustrations drawn from arithmetic. But this point of view really is just about as widespread and traditional in geometry as in arithmetic. Thus it is common to cite such geometric propositions as the Pythagorean theorem ("in a right triangle the square of the hypotenuse equals the sum of the squares of the other two sides"), and the theorem concerning the angle sum of a triangle ("the sum of the angles of a triangle equals one straight angle, or 180°"), as illustrations of the absolute finality of geometric knowledge.

If someone challenges the absoluteness of these theorems the typical reply is that there can be no question about it, because ever since the time of Euclid (300 B.C.), or earlier, it has been known that these and other geometric statements are actually *provable* without any guesswork. If the challenger then asks on what basis the proofs rest, the stock answer is "the proofs depend entirely on logical reasoning applied to self-evident truths." These supposedly self-evident truths are the axioms and postulates of geometry. The supposedly complete reliance on logical reasoning is thought to guarantee a method of arriving at unquestionably true results from the self-evident axioms and postulates.

The only trouble with this comfortably reassuring position in geometry is that it has been very thoroughly exploded by significant events in the history of mathematics and science! These significant events were principally the following: (1) the rise and mathematical development of the non-Euclidean geometries during the 19th century; (2) studies in the postulational and logical foundations of geometry in general (with considerable emphasis on Euclidean geometry and deficiencies in Euclid's logical rigor), beginning in the latter part of the 19th and continuing into the 20th century; (3) the formulation of the Einstein theory of relativity during the early part of the 20th century. All of these events will be important for us in a number of different connections in this text. Here we wish to consider primarily the impact of event number (1) or, let us say, the existence of non-Euclidean geometries, on the absolute truth point of view in geometry.

First, let us clarify the terminology. Euclidean geometry, of course, refers to the general type of geometry first logically organized (as far as we know) by Euclid. This is the type studied in the standard high school courses in plane and solid geometry. On the

other hand, there are two main varieties of non-Euclidean geometries, known respectively as *Lobachevskian* (or *hyperbolic*) geometry, and *Riemannian* (or *elliptic*) geometry, after the mathematicians Lobachevski and Riemann, credited with founding them.*

Although these two standard non-Euclidean geometries disagree in certain important respects with Euclidean geometry, it should be noted that they are not by any means the only systems of geometry which differ from Euclid's system. They are simply the ones which are historically significant, and for which the name "non-Euclidean" is usually reserved. Furthermore, they are certified by mathematicians to be just as sound logically as Euclidean geometry.

In order to emphasize the bearing of non-Euclidean geometries on the absolute truth point of view, let us consider the familiar Euclidean theorem about the angle sum of a triangle, which we have mentioned above as a geometric counterpart of "$2 \times 2 = 4$" in arithmetic. The Euclidean theorem means that the sum of the angles of a triangle is *exactly 180°*, neither more nor less by one-millionth of a second. No question of *actual measurement* of the angles of a physical triangle, drawn on paper or determined by surveyors' stakes, is involved; we know that the theorem refers to theoretical or mathematical triangles.

We now exhibit for comparison with the Euclidean angle sum theorem the corresponding theorems of the Lobachevskian and Riemannian geometries:

Euclidean theorem. The sum of the angles of a triangle *is equal to* 180°.

Lobachevskian theorem. The sum of the angles of a triangle *is less than* 180°.

Riemannian theorem. The sum of the angles of a triangle *is greater than* 180°.

Thus the conflict here among the three geometries is clear-cut.†

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* Lobachevski (1793–1856) was a Russian geometer who began expounding his system of geometry about 1830. The name of a Hungarian mathematician, Bolyai, is also associated with the Lobachevskian geometry. Riemann (1826–1866) was a German mathematician of wide interests, whose contribution to the other system of non-Euclidean geometry, dating from his doctoral dissertation at Göttingen about 1850, was from the more advanced point of view of differential geometry.

† The degree as a unit of angular measure is the same in all three geometries. In other words, in each theorem we can read "one straight angle" instead of 180°.

The question naturally arises as to how equal validity is possible for such mutually contradictory statements. To answer this question it is desirable to recall, first, the general nature of proofs of theorems in geometry and, second, a typical proof used for the *Euclidean* angle sum theorem.

It is easy to see that in geometry, or any logically organized branch of mathematics, the proof of a specified theorem depends in the last analysis on those assumptions (postulates or axioms) and definitions used either directly or indirectly (through the intermediary means of previous theorems) in the statements making up the proof.

Next we inquire what the basis of proof is for the Euclidean angle sum theorem. The customary procedure in proving this theorem in high school plane geometry is to draw a construction line (DE) through the vertex (B) of the triangle, parallel to the base (AC) (see Fig. 1–1). Then the three angles of the triangle are shown to be equal respectively to the three angles at the vertex B on one side of the constructed line. Thus the sum of the angles of the triangle must be equal to one straight angle. The proof seems innocent enough but it cannot be valid in the non-Euclidean geometries. Why not?

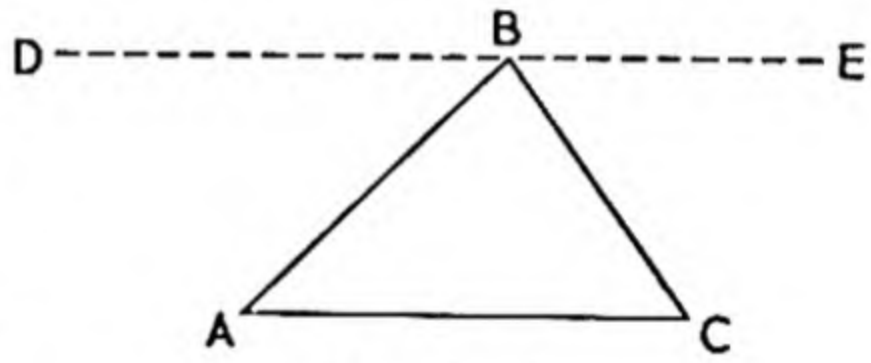

Fig. 1–1

The only possible explanation is that certain fundamental assumptions or definitions used in this proof are not valid in non-Euclidean geometries. It is not hard to guess that the source of the trouble lies in the parallel construction line, or its use for the purpose of proving that certain required alternate interior angles are equal. The exercise below will be helpful in completing this discussion.

Exercise

High school geometry textbooks usually include a postulate, often called the "Euclidean parallel postulate," to the effect that "through a point not on a line there is one, and only one, line parallel to the given line." Both parts of this postulate are really used in the proof outlined above for the angle sum theorem. Show in detail how each part is involved. (This will require investigating the basis of proof of a "previous theorem" also.)

1–6 The postulates and other properties of the three geometries. We are now in a position to give a brief explanation of the reason for

conflicting theorems concerning the sum of the angles of a triangle in the Euclidean and non-Euclidean plane geometries. Here we omit a number of refinements, but shall give a few more details in footnotes.

The explanation is essentially as follows. The crucial postulate used in the proof of the Euclidean angle sum theorem — namely, the Euclidean parallel postulate — is contradicted by the postulates about parallel lines adopted in the non-Euclidean geometries. The three postulates in question are listed here in similar form for purposes of comparison (see Fig. 1–2):

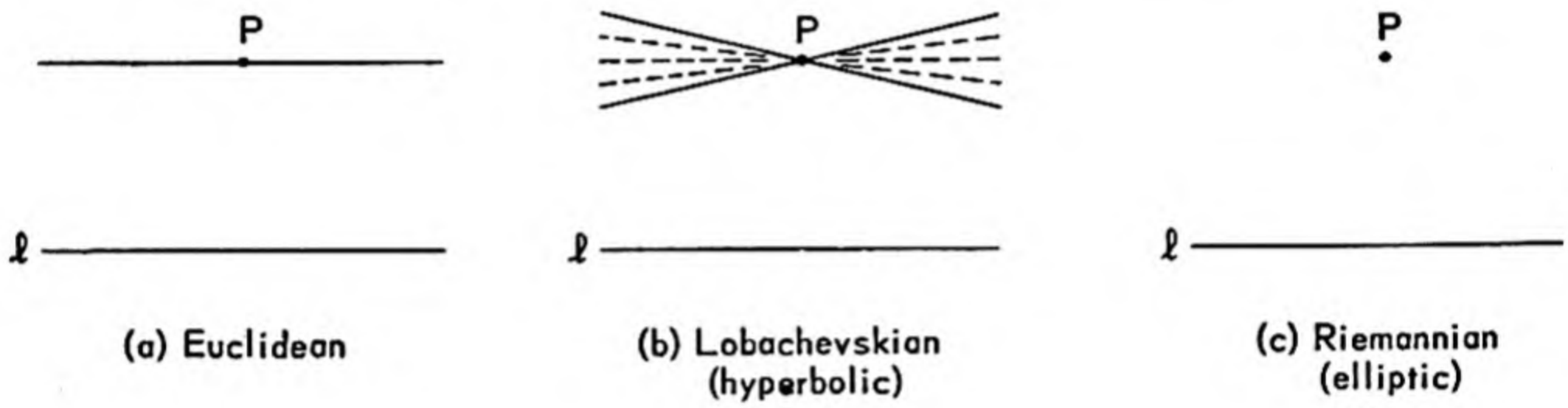

Fig. 1–2

*Euclidean parallel postulate.** Through a point not on a given line there is *one and only one line parallel* to the given line.

Lobachevskian parallel postulate.† Through a point not on a given line there are *two lines parallel* to the given line.

Riemannian parallel postulate. Through a point not on a given line there is *no line parallel* to the given line.

In Lobachevskian (hyperbolic) geometry the other postulates remain the same as in the Euclidean case. In Riemannian (elliptic) geometry, to avoid contradictions a few additional changes have to

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* This is not the actual statement of Euclid's own famous "postulate of parallels." The latter involves the concept of a transversal of two lines, and paradoxically does not make use of the word "parallel." Euclid's postulate will be presented and discussed briefly in Sec. 2–4.

† Referring to Fig. 1–2 (b), we note that if two lines through P are parallel to line l, then infinitely many lines through P (such as the dotted lines in the figure) will fail to intersect l. The word "parallel" in Lobachevskian geometry is usually reserved for the bounding lines between those lines through P which intersect and those which fail to intersect l. The other lines through P which fail to intersect l can be defined to be "ultraparallel" to l.

be made in postulates assumed either tacitly or explicitly in Euclidean geometry. For example, in Riemannian geometry, a straight line is assumed to be unbounded but is not assumed infinite in length; two straight lines may enclose an area; and there may possibly be more than one straight line joining two distinct points.

Inasmuch as the Euclidean parallel postulate plays such a crucial role in the proof of the Euclidean angle sum theorem, it is hardly surprising that changes made in this postulate will lead to changes in the angle sum theorem.

Of course, there are certain theorems of Euclidean geometry which depend only on postulates common to all three geometries. Accordingly, these theorems will be common to all of the geometries. In this category, for example, are the standard theorems on congruent triangles, isosceles triangles, and relations of sides opposite unequal angles in a triangle (and *vice versa*).

There will be a still larger body of theorems which are common to Euclidean and Lobachevskian geometry, but which do not all hold in Riemannian geometry. These include all of the theorems proved in high school plane geometry textbooks prior to the introduction of parallel lines. One of these, which does not hold in Riemannian geometry, is the theorem stating that the exterior angle of a triangle is greater than either opposite interior angle, for the proof of this theorem requires a tacit assumption that a straight line is infinite in length. Actually, it is provable in Riemannian geometry that all straight lines in the plane are closed paths and are of a finite constant length. Furthermore, all the perpendiculars to a given straight line meet either in a single common point or in two distinct points, depending on the particular subdivision of Riemannian geometry under consideration.*

For a list of characteristic distinguishing properties of the three geometries, a table is included at the end of this section. Here, however, our main object is to emphasize the distinction between the geometries as based on the various postulates about parallel lines and corresponding theorems about the angle sum of a triangle.

The objection which most persons have to the non-Euclidean

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* There are two varieties of Riemannian plane geometry; they are known as *single elliptic* and *double elliptic* geometry. In the first variety there is one and only one straight line through two distinct points. In the second variety there exist infinitely many straight lines through certain pairs of points.

parallel postulates when they see them for the first time is to the effect that they are absurd or impossible, usually meaning contrary to common sense. Three considerations seem pertinent in reply to this objection.

(1) History provides many examples of serious mistakes due to reliance on common-sense judgments. We need only refer to numerous outdated teachings in astronomy, geography, physics, chemistry, medicine, and other sciences.

(2) Contradiction of common sense is not the same as logical contradiction. A technical device (which will be considered in Section 11–2) shows that the non-Euclidean geometries are just as free from logical contradiction as Euclidean geometry is. The same kind of logical reasoning is used in both. Hence, from the standpoint of pure mathematics both the Euclidean and non-Euclidean geometries constitute equally sound geometric doctrines (using the word "doctrine" in the sense of a theory or logically organized structure of thought).

(3) Again from the standpoint of pure mathematics, it does not help to claim that the non-Euclidean postulates contradict our physical experience regarding parallel lines. Mathematically, parallel lines, like triangles, circles, and all other geometric concepts, are idealized abstractions which cannot actually be identified with their physical counterparts, such as pencil marks on paper, chalk marks on blackboard, surveyors' "lines of sight," or astronomers' light rays. Hence parallel lines cannot strictly be subjected to experimental observations by the use of drawing or surveying instruments. In any case, there is no logical compulsion to be guided by experimental evidence in constructing a *pure* geometry (although this may often be desirable).

The following is a popularized version of the device mentioned in item (2) above, for the case of Riemannian plane geometry. Imagine a world populated by intelligent beings who are restricted entirely to the surface of a large sphere in ordinary three-dimensional Euclidean space. They are led to construct a plane geometry to describe their space, which seems "flat" to them (just as in the Middle Ages the earth was asserted to be flat). "Straight lines" or lines of shortest distance are observed by them to exist on their "plane." These correspond to paths known to observers from the outside as *great circles* on the sphere. (A great circle is the circle in which a plane through the center of a sphere cuts the sphere. Familiar examples are the meridians and equator of the earth.) But any two great circles on a

sphere intersect (in two points). Furthermore, all great circles are unbounded and have a constant finite length. These are the properties which would be the natural characteristics of "straight lines" for our intelligent beings. They would willingly adopt as a postulate the common-sense fact that no parallel "lines" exist. And they might be expected to prove that the sum of the angles of a triangle is greater than 180° — inasmuch as one of their "triangles" seen by an outside observer would be a *spherical triangle* (formed by arcs of three great circles), and it is a theorem of solid Euclidean geometry that the sum of the angles of a spherical triangle is greater than 180°.

The upshot of this illustration is that the type of plane geometry constructed by the intelligent beings would be Riemannian (double elliptic) plane geometry, which is thus seen to be *logically* and *abstractly* analogous to Euclidean geometry on the surface of a sphere. Hence any contradiction in Riemannian plane geometry would be reflected in a corresponding contradiction in Euclidean spherical geometry. This same example will be presented from a more advanced postulational point of view in Section 11–2.*

If now we pass from *pure* mathematics to *applied* mathematics, then the point of view shifts somewhat and we consider the three *theoretical* systems of geometry as possible competing descriptions (although admittedly idealized and hence at best approximate descriptions) of the real physical space in which we live. Which one of these offers the *best* description is a problem for the physicist or astronomer to decide. At the present stage of development of physical and astronomical knowledge, this question seems to have no clear-cut answer. The problem will be considered again later in the text when we discuss geometry in relation to physical science (see Section 5–6).

If someone is curious about possible practical applications of the non-Euclidean geometries on a more restricted scale, we mention a recent combined mathematical and psychological study in binocular vision.† The author of this study made extensive use of the non-

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* For similar illustrations in the case of Lobachevskian geometry see Cooley, Gans, second edition, Chapter 22, or first edition, Chapter XVIII; or R. Courant and H. Robbins, *What is Mathematics?*, Oxford University Press, 1941, pp. 218–224.

† Rudolf K. Luneburg, *Mathematical Analysis of Binocular Vision* (published for the Dartmouth Eye Institute), Princeton University Press, 1947.

SOME CHARACTERISTIC PROPERTIES OF EUCLIDEAN AND NON-EUCLIDEAN PLANE GEOMETRIES*

Euclidean	*Lobachevskian (hyperbolic)*	*Riemannian (elliptic)*
1. There is [one and] *only one* line through a point parallel to a given line.	1. There are *two* lines through a point parallel to a given line [and infinitely many ultraparallels].	1. There is *no* line through a point parallel to a given line.
2. A straight line is unbounded and has an infinite length.	2. A straight line is unbounded and has an infinite length.	2. A straight line is unbounded. [It has a finite total length and all straight lines have the same total length.]
3. There is one and only one straight line through two given points.	3. There is one and only one straight line through two given points.	3. There is at least one line through two given points. [Through certain pairs of points there exist infinitely many straight lines.†]
4. The sum of the angles of a triangle equals 180°. (The angle sum of a triangle is constant, regardless of the area of the triangle.)	4. The sum of the angles of a triangle is *less than* 180°. (The angular defect, i.e., the discrepancy from 180°, is proportional to the area of the triangle; hence the angle sum of a triangle approaches 180° as its area approaches 0.)	4. The sum of the angles of a triangle is *greater than* 180°. (The angular excess, over 180°, is proportional to the area of the triangle; hence the angle sum of the triangle approaches 180° as its area approaches 0.)
5. Similar triangles, or polygons, of different areas exist.	5. Similar triangles, or polygons, of different areas cannot exist.	5. Similar triangles, or polygons, of different areas cannot exist.
6. Two parallel lines are everywhere equidistant.	6. Two parallel lines are not equidistant; they approach each other in one direction and diverge in the other.	6. ——
7. Two lines perpendicular to the same line are parallel.	7. Two lines perpendicular to the same line are ultraparallel.	7. Two lines perpendicular to the same line intersect; and all lines perpendicular to the same line have a common point of intersection (or two such points †) at the same distance from the given line.
8. Two parallel straight lines have infinitely many common perpendiculars.	8. Two parallel or intersecting lines have no common perpendicular; two ultraparallel lines have a unique common perpendicular.	8. Any two (intersecting) lines have a unique common perpendicular.
9. Rectangles (in the sense of parallelograms with *all angles right angles*) exist.	9. Rectangles do not exist.	9. Rectangles do not exist.
10. The Pythagorean Theorem: The sum of the squares of two legs of a right triangle equals the square of the hypotenuse.	10. The Pythagorean Theorem does not hold but becomes more nearly true as the area of a right triangle approaches 0.	10. The Pythagorean Theorem does not hold, but becomes more nearly true as the area of a right triangle approaches 0.

* In this list, properties 1, 2, and 3 in each column are to be considered as fundamental postulates, with the exception of the bracketed portions, which are provable. Properties 4 to 10 in each case are theorems.

† A special property of double elliptic geometry.

Euclidean geometries (from the advanced point of view of differential geometry) and reached the conclusion that the *best* geometry to describe *visual space* (i.e., space as psychologically observed by persons with normal vision) is the Lobachevskian, or hyperbolic, geometry.

Even without considering practical applications to visual space or physical space, however, the great value of the non-Euclidean geometries in the development of mathematical and physical thought can hardly be denied. For they have contributed to a better understanding of the postulational method in mathematics and its significance for science than would have been possible if mathematicians, physicists, and astronomers still felt obliged to accept Euclidean geometry as a revelation of ultimate geometric truth.

Exercises

1. Give examples of "outdated teachings" in several of the specific fields mentioned above, or in some other science. Try to find cases in which common sense or intuition has been superseded by less obvious judgments.

2. In building up a *theoretical* system of *pure* geometry, why is it impossible to establish the necessity for any postulate by *experimental* or observational means? Consider simple postulates, e.g., "there is one and only one line through two distinct points," as well as the parallel postulate. Also consider such questions as the meaning of a *mathematical* point, line, or figure.

3. Do you think it would conceivably be possible to give a proof of the angle sum theorem in Euclidean geometry without use of the Euclidean parallel postulate or some equivalent postulate (assuming that all other Euclidean postulates are retained)?

4. In all three geometries a special quadrilateral can be constructed by starting with a given base, erecting two segments of equal length perpendicular to it, and then connecting the end points of the perpendiculars. The resulting figure is known as an *isosceles birectangular quadrilateral.*

It can be shown (by congruent triangles) that in all three cases the "summit angles," i.e., the angles formed by the upper base and the equal legs, will be equal. In the Euclidean case, of course, these will be right angles and the figure is a rectangle. What will the angles be (a) in Lobachevskian geometry and (b) in Riemannian geometry?

5. What experimental test could you suggest for deciding which of the three geometries offers the best description of physical space? Would the test be likely to be conclusive?

6. Investigate the historical background and development of the non-Euclidean geometries. (A good reference for this purpose is H. E. Wolfe, *Introduction to Non-Euclidean Geometry,* Dryden Press, 1945, Chapter III.

See also R. Courant and H. Robbins, *What is Mathematics?*, Oxford University Press, 1941, pp. 218–220; or Cooley, Gans, second edition, Chapter 22.)

7. Investigate the logical development of the theorems leading up to the Lobachevskian angle sum theorem. (See Richardson, Chapter XVI. See also the monograph "Non-Euclidean Geometry" by F. S. Woods, in *Monographs on Topics of Modern Mathematics*, edited by Jacob W. A. Young, Longmans, Green and Co., 1927, pp. 93–147.)

1–7 Mathematical problems, solved and unsolved. Returning to the traditional popular view of the finality of mathematical knowledge and of the absoluteness of the results arrived at by mathematicians, we note that there is one familiar exception to or qualification of this view. It is commonly believed that mathematicians have shown themselves to be professionally incompetent and lacking in ingenuity, indeed that they have failed completely in connection with the three famous construction problems of ancient Greek geometry.

These problems are (1) *trisecting the angle*, (2) *duplicating the cube*, and (3) *squaring the circle*. In their original form, these were all proposed as constructions to be performed (as in high school plane geometry) with ruler and compasses alone ("ruler" meaning an unmarked straightedge used solely for the purpose of drawing straight lines).

The trisection problem is practically self-explanatory, namely: *given an arbitrary angle, to construct an angle equal to one-third of the given angle.*

The problem of duplicating the cube is: *given a cube, to construct the edge of a new cube whose volume will be exactly twice the volume of the given cube.*

The problem of squaring the circle is: *given a circle, to construct the side of a square whose area will be exactly equal to the area of the circle.*

One of the frequent pastimes of amateur mathematicians is to produce elaborate constructions for trisecting an angle or squaring the circle, under the impression that the problems have not been solved. The fact of the matter is, however, that in a very real sense mathematicians *have solved* all three of the famous Greek problems.

To explain the apparent paradox it is sufficient to note that a construction problem represents a task to be attempted under certain specified preliminary conditions, on the tentative assumption that the task *can* be performed under those conditions. In the standard cases a construction problem is solved by producing a construction

and proving that it is correct. This provides the best possible argument that the construction can be done as specified. On the other hand, in certain situations it may turn out that the assigned task cannot be performed under the prescribed conditions. Certainly if it can be proved that such is the case, the problem is disposed of and can be considered solved, for there is no longer any use in attempting to perform the desired constructions.

It is in this sense that mathematicians have solved, and solved convincingly, the three famous problems. By use of theory of equations, or still higher branches of algebra, together with elementary Euclidean geometry and analytic geometry, criteria for ruler-and-compass constructibility have been derived. These criteria, although they depend on more than one branch of mathematics, should carry as much weight in truth value as any theorem of Euclidean or non-Euclidean geometry. The criteria show that a desired construction is possible if and only if the construction, when formulated algebraically, leads to a certain special type of equation. It has been proved that the constructions of trisecting the angle, duplicating the cube, and squaring the circle all fail to produce the required type of equation, and thus the constructions are known to be impossible. This negative result is not at all an unusual situation mathematically.

Someone may object that it is perfectly possible to trisect an angle of 180°, or 90°, or 45°, or many other special angles. This is true, but irrelevant. For in the problem of trisecting an angle, as stated above, the given angle is *arbitrary*, meaning of no specified size. Conceivably (although this is far from the case), we might be able to trisect every angle of a whole number of degrees from 1° to 180°, but still this would not prove that an arbitrary angle can be trisected.

Again, it may be asked, how can we account for the numerous trisections produced (and sometimes published in newspapers and magazines) by amateurs? The answer is that the constructions quite probably involve some kind of special case, or else fallacious methods are employed in the construction or proof. Often a ruler may be used in an illegitimate manner, such as with marked points on it, or it may be used by sliding instead of just for drawing straight lines. Actually there is a well-known method of Archimedes for performing a makeshift trisection using a ruler in this disallowed way.

Many methods, other than the strict ruler-and-compass procedure, were devised by other Greek mathematicians for performing all of the desired constructions. These involved the use either of familiar

curves such as the parabola and hyperbola (for duplicating the cube), or specially invented curves such as the trisectrix, conchoid, or quadratrix (for angle trisection and squaring the circle). Of course, all of these devices were failures with reference to the problems as originally formulated but nonetheless they contributed to an enrichment of mathematical knowledge. It was not until the latter part of the 19th century that all of the problems were settled in the sense already described. The culmination came with the valuable proof (by the mathematician Lindemann about 1880) that the number π is *transcendental* * and hence cannot satisfy an equation of the type required by the constructibility criterion. This showed that the proposed construction of squaring the circle is impossible.

Finally, it should be pointed out that there are numerous, long investigated, and still open problems in mathematics which are historically famous. Among these are the problem of Fermat's Last Theorem, the four-color map problem, Cantor's continuum problem, and many problems about prime numbers. We shall not discuss these here, but suggest them as possibilities for supplementary investigation.

That such unsolved problems exist can hardly detract from other known mathematical results, and can hardly be considered degrading to mathematical knowledge. Indeed, just as in the case of scientific research, much mathematical research originates from the stimulus of unsolved problems, and advancement of knowledge almost always results whether or not success is achieved in solving the specified problems.

One further thing — recent work of a fundamental nature in mathematical logic makes it quite conceivable that some of the unsolved problems might eventually be proved to be unsolvable, or "undecidable" in the sense of an indicated impossibility of reaching either a positive or negative answer to the problem. But this in itself would be a step forward.

EXERCISES

1. Give other examples of negative results (asserting that something is impossible): (a) in algebra, (b) in Euclidean geometry, (c) in trigonometry.
2. Investigate Archimedes' makeshift trisection. (See one of the follow-

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* A *transcendental* number is a real number which cannot be the root of an algebraic equation with integral coefficients, i.e., an equation of the form $a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$, where the a's are integers.

ing: D. E. Smith, *History of Mathematics*, Vol. II, Ginn & Co., 1925, p. 300; L. E. Dickson, *New First Course in the Theory of Equations*, John Wiley, 1939, p. 40.)

3. Investigate some special curves that have been used for angle trisection, etc. (See D. E. Smith, *History of Mathematics*, Vol. II, pp. 298–316; H. T. Davis, *College Algebra*, Prentice-Hall, 1940.)

4. Investigate the problem of constructing regular polygons of a given number of sides. (See Richardson, Chapter VIII, or Dickson, *New First Course in Theory of Equations*, Chapter XII.)

5. Investigate some of the genuine unsolved problems in mathematics. (See Richardson, Chapter VIII.)

1–8 Conclusion. The relative nature of mathematical truth. The purpose of this chapter has been to show the need of revision of the popular point of view that mathematical knowledge consists of absolute facts and unalterable truths, and that mathematicians "know all the answers."

We have seen that such a seemingly absolute fact as $2 \times 2 = 4$ must be considered to be either an unproved assumption, or else a theorem depending on other assumptions and definitions.

We have given illustrations of numerical situations where it is natural and useful to employ a type of arithmetic different from the usual. This is accomplished by a slight change in the meaning of symbols used, as in our examples for adding days of the week and in the various systems of modulo arithmetic. Some of the properties of these systems turn out to be significantly different from ordinary arithmetic.

Differences of a less significant nature, but nevertheless spectacular at first sight, were illustrated by considering a change of number base from ten to some other number such as five, eight, or twelve.

We have also mentioned the existence and good mathematical standing of the non-Euclidean geometries. We have compared the Euclidean and non-Euclidean geometries, with special attention to the conflicting theorems about the sum of the angles of a triangle in these geometries, and the conflicting postulates about parallel lines which are largely responsible for the conflicting theorems. We have seen that we are no longer entitled to view the theorems of Euclidean geometry as absolute truths, but rather as statements which are true *if* the Euclidean postulates are true, and similarly for the non-Euclidean theorems.

In connection with the three famous construction problems of

Greek geometry we have found that the popular view is mistaken in the opposite direction from usual. The accusation is, ironically, that mathematical knowledge is deficient in definiteness here. Actually we know that the problems have been completely solved, and the result can be formulated as a theorem stating the impossibility of performing the constructions under the specified conditions. Such a theorem has as firm a foundation and proof as any theorem of Euclidean geometry or any rule of arithmetic.

The whole moral, of course, is that the absolute truth point of view must be discarded. The "facts" of any branch of pure mathematics must be recognized as being assumptions (postulates or axioms), or definitions, or theorems. The lesson of non-Euclidean geometry shows the danger of considering geometric postulates as "self-evident truths." Even in the case of the assumptions that are usually labeled as "axioms" in plane geometry and algebra (for example, statements about equality and inequality), this is dangerous. (In the study of infinite classes and transfinite numbers it turns out that certain familiar axioms have to be discarded. See the exercises below, inserted as a postscript to this chapter.)

The safest view to adopt, subject to elaboration later on in the text, is that the assumptions are preliminary statements which form a starting point for the logical development of the branch of mathematics in question. Axioms and postulates become more or less indistinguishable from this point of view, and mathematicians often use the two designations interchangeably. Hereafter we shall tend to use the word "postulate" for any officially stated assumption. As for the theorems, they should be viewed merely as what they are: logical consequences of the postulates and definitions. The most that can be claimed is that *if* the postulates are true *and* the definitions are accepted, and *if* the methods of reasoning are sound, *then* the theorems are true. In other words, we arrive at a concept of *relative truth* (of theorems in relation to postulates, definitions, and logical reasoning) to replace the *absolute truth* point of view.

EXERCISES

A basic concept in many branches of mathematics (from the elementary school to the graduate school level) is that of a *set*, or *class*, or *collection* of objects. The objects may be as tangible as chairs or pieces of chalk, or as abstract as idealized mathematical points or complicated types of numbers. The individual objects are known as *members*, or *elements*, of the set. A set

is finite if its members can be *counted* completely in such a way that the counting process terminates. If this cannot be done, the set is infinite.

A terminated counting process is essentially the process of *pairing* the elements of a given set with a *subset* of the natural numbers, taken in order, starting with 1 and *ending* with a definite natural number (e.g., 17, 1003, 10,000,000). The pairing establishes what is known as a *one-to-one* (1–1) *correspondence* between the elements of the given set and the subset of natural numbers.

On the other hand, it may be possible to devise a nonterminated counting process for an infinite set in such a way that a complete 1–1 correspondence is established between elements of the given set and the set of *all* natural numbers (positive integers). In this case we say the given set is *denumerably infinite.*

For example, imagine the set of *all* positive integers written on one line and the set of their squares written on another line in corresponding positions:

(a)	1	2	3	4	5	6	.	.	. .
(b)	1	4	9	16	25	36	.	.	. .

Line (a) provides a systematic way of *counting* the perfect square integers in line (b). This gives a proof that the set of perfect square integers is denumerably infinite.

From another point of view it is reasonable to say that the infinite sets represented by lines (a) and (b) have the *same number* of elements. Paradoxically, the elements listed in line (a) include all of those in line (b) [the perfect squares] and others besides [the nonsquare integers]. Thus, we have an apparent breakdown of the axiom that the whole is greater than any of its parts.

1. Now show that the following sets are denumerably infinite: (a) the set of all even integers, (b) the set of all odd integers.

2. (a) Investigate a proof that the set of all rational numbers is denumerably infinite. (b) Investigate a proof that the set of all real numbers is nondenumerably infinite. (See Richardson, Chapter XV; or Cooley, Gans, second edition, Chapter 21, or first edition, Chapter XVII.)

CHAPTER 2

THE ORIGIN AND INFLUENCE OF LOGICAL SYSTEMS

2–1 Introduction. The concept arrived at in the preceding chapter, of the relative rather than absolute truth of mathematical theorems, is a natural outgrowth of the mathematical plan of logical organization on a postulate-definition-theorem basis. The structure of thought built up on such a plan may be called a *logical system*, or alternatively a *postulational* or *deductive system*, or a *deductive theory*.

The present chapter will be concerned with logical systems and their use and value both in mathematics and in other departments of intellectual activity. We begin with historical background material by comparing briefly the contributions of the ancient Egyptian and Greek civilizations in mathematics, especially geometry.

2–2 Egyptian and Greek mathematics. The ancient Egyptians apparently had a remarkable practical knowledge of geometry which they put to use in building temples and pyramids, and in surveying land boundaries. But they seem to have had little feeling for the abstract side of geometry. It was here that the Greeks excelled. Greek research in geometry, and in mathematics in general, was notable for its emphasis on the logical development of pure subject matter and on investigations of theoretical problems (such as the famous construction problems discussed in Chapter 1), usually without immediate concern for practical applications. No doubt Egyptian geometers would have considered most Greek geometry *useless*. Nevertheless, the verdict of time certainly favors the mathematical contributions of the Greeks over those of the Egyptians.

It is easy to account for this verdict. Is it not largely that the Greek mathematicians had a broader concept of what is *useful* than the Egyptians had? The Greeks considered abstract intellectual activity to be essentially an end in itself, requiring no further justification. Thus they were led to develop ideals of logical organization, as exemplified by the works of Euclid, Apollonius, and Archimedes. These men produced systematic mathematical treatises which served as guides for many future generations. Thus many years later Newton applied Euclid's system of geometry in his theory

of the universe; and both Kepler and Newton made good use in their astronomical research of the abstract principles of conic sections logically developed by Appollonius.

Entirely aside from subject matter, the Greek plan of logical organization itself turned out to be a contribution of the first magnitude. It is this plan which is our primary concern now, and we continue with a discussion of Euclid as its most influential exponent historically. We shall include some criticism of Euclid from the standpoint of modern logical rigor.

2–3 Euclid and the concept of a logical system: definitions. Euclid (*c.* 300 B.C.) was probably the first Greek mathematician to set forth a large body of mathematical knowledge in the form of a well-developed logical system. His reputation rests largely on his treatise, *The Elements*, which was used for centuries (in many different editions) as the world's authoritative textbook in theoretical geometry. Euclid's main purpose in the *Elements* was the systematic deduction, by logical proof, of theorems and constructions from a small number of stated axioms and postulates and, of course, definitions.

Contrary to popular impression, the *Elements* is not restricted to the field of geometry. It also contains subject matter which would be classified today as higher arithmetic or theory of numbers, together with a study of what we would call algebraic identities. However, it is true that Euclid's point of view was predominantly geometric, and the influence of his sequence of topics in geometry is still apparent in modern textbooks for high school geometry.

We shall not make an extensive study of Euclid's *Elements*, but instead shall concentrate on certain characteristics of the treatise which will be useful in discussing the concept of a logical system.

Euclid's Book I starts with a list of technical definitions which are to be used later in the axioms, postulates, and theorems. The following are the first four definitions:

1. A point is that which has no parts, or which has no magnitude.
2. A line is length without breadth.
3. The extremities of a line are points.
4. A straight line is that which lies evenly between its extreme points.

Do these preliminary definitions of Euclid actually serve to tell a reader the *meaning* of the words "point," "line," "extremities,"

"straight line"? If this is so, then the reader must be expected to know in the case of the definition of *point*, the meaning of the following other crucial words "that," "part," "magnitude." In the definition of *line*, the reader must be expected to understand the words "length and breadth," and in the case of *straight* line, "lies evenly between." Not only are these other words not defined, but no official mention is made of this fact.

Would it have been possible for Euclid to define explicitly all the technical terms which he used? For that matter, would it be possible to do this in any formal series of definitions? In any such series there would have to be a beginning somewhere, a *first* definition. And the other words used in this definition (aside from words used as in ordinary language) would have to be considered as *undefined*, because it would not be proper to make use of a technical term defined in one of the later definitions.

Nowadays, the need for undefined words in any logical system is as well recognized as the need for postulates or unproved propositions, and for analogous reasons. Furthermore, it is considered desirable to adopt an official list of undefined terms in addition to an official list of postulates. Euclid realized the desirability of the latter but overlooked the equal desirability of the former. From our standpoint, this is a defect in his logical organization.

2–4 Euclid's postulates, axioms, and propositions. Next we turn to Euclid's *postulates* and *axioms*. Different versions of the *Elements* differ as to the listing of the postulates and axioms, but according to the outstanding modern authority on Euclid, Sir Thomas L. Heath,* there were five axioms and five postulates. The axioms were "common notions" (including, for example, "things equal to the same thing are equal to each other"), while the postulates were specifically geometric in nature.

For reference we state Euclid's five postulates:

POSTULATE 1. A straight line may be drawn from any one point to any other point.

POSTULATE 2. A terminated straight line may be produced to any length in a straight line.

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* See *The Elements of Euclid* (with introduction by Sir Thomas L. Heath), Everyman's Library, E. P. Dutton, 1933.

POSTULATE 3. A circle may be described from any center at any distance from that center.

POSTULATE 4. All right angles are equal to one another.

POSTULATE 5. If a straight line meets two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles.

Of these postulates, the fifth, known as Euclid's "postulate of parallels," is by far the most famous and in itself forms an important topic in the historical background of non-Euclidean geometry.* This postulate has been both praised for its ingenuity and criticized for its lack of "self-evidence" in comparison with the other postulates.

The fifth postulate is essentially equivalent to one half of what we referred to in Section 1–6 as the *Euclidean* parallel postulate—namely, the part which says that there is *only one* parallel to a given line through an outside point. Euclid proved (or thought he proved) the existence of *one* such parallel by making use of a previous theorem to the effect that an exterior angle of a triangle is greater than either opposite interior angle. But in proving the latter he tacitly assumed that a straight line is infinite in length.† Sometimes Postulate 2 is claimed to justify this, but such an interpretation seems to add something new to what Euclid said. In a somewhat similar manner Postulate 1 does not convey all that Euclid required — namely, that there is one *and only one* line through two points. Thus we have a preliminary indication that Euclid's postulates were not sufficiently precise or complete for his purpose.

Euclid's ideal was to deduce all of his propositions from the official postulates and axioms without recourse to any unofficial assumptions, or appearances of diagrams. This is also the ideal in any high school course in demonstrative plane geometry, or in any logical system. From this standpoint it is known that there are numerous logical defects in the *Elements* (and likewise in high school plane geometry).

As a more detailed example let us consider Euclid's Proposition 1

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* See, for example, H. E. Wolfe, *Introduction to Non-Euclidean Geometry*, Dryden Press, 1945, Chapter II.

† See Wolfe, *op. cit.*, p. 8.

of Book I (commonly designated as "Euclid I.1"), a construction problem: "To describe an equilateral triangle on a given finite straight line." The procedure as given in the Everyman's edition of Euclid * is as follows (referring to Fig. 2–1):

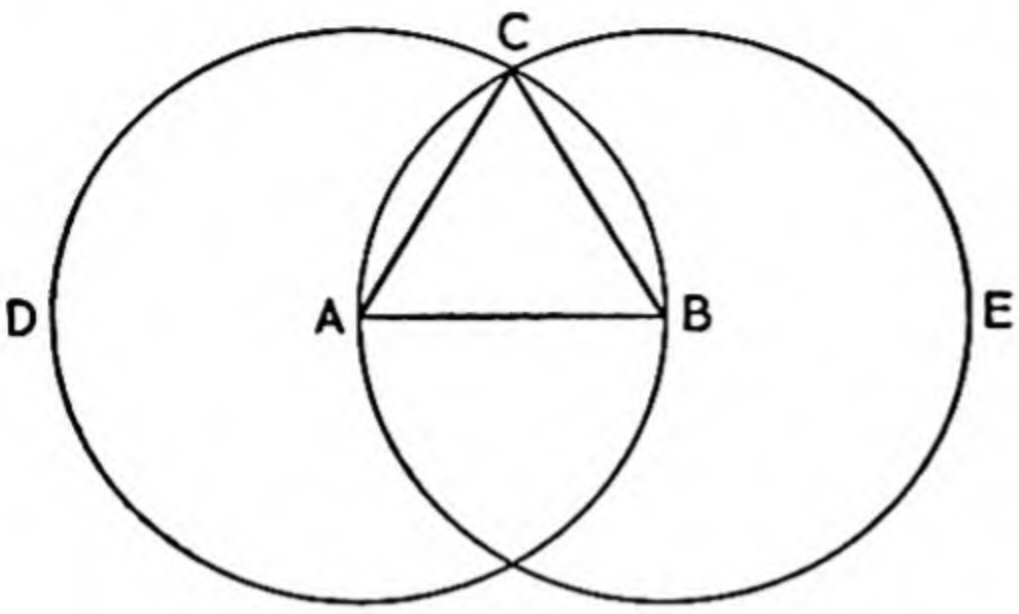

Fig. 2–1

"Let AB be the given straight line: it is required to describe an equilateral triangle on AB.

From the centre A, at the distance AB, describe the circle BCD. [Postulate 3.

From the centre B, at the distance BA, describe the circle ACE. [Postulate 3.

From the point C, at which the circles cut one another, draw the straight lines CA and CB to the points A and B. [Postulate 1.

ABC shall be an equilateral triangle.

Because the point A is the centre of the circle BCD, AC is equal to AB. [Definition 15.

And because the point B is the centre of the circle ACE, BC is equal to BA. [Definition 15.

But it has been shown that CA is equal to AB; therefore CA and CB are each of them equal to AB.

But things which are equal to the same thing are equal to one another. [Axiom 1.

Therefore CA is equal to CB.

Therefore CA, AB, BC are equal to one another.

Wherefore the triangle ABC is equilateral, [Definition 24.
and it is described on the given straight line AB. Q.E.F.†"

The defect in the proof of "Euclid I.1" (quoted above) occurs in connection with the point C, one intersection point of the constructed circles. The trouble is, how do we know there is such a point?

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* *The Elements of Euclid, op. cit.*, p. 7.

† Q.E.F. used at the end of a construction means *quod erat faciendum*, i.e., "which was to be done."

Euclid simply assumes without any proof that the "circles cut one another" at C and proceeds to use C in the construction and proof. We find nothing in the postulates which justifies this procedure. It is insufficient answer to say that the circles *have* to intersect at some point. Granted, it seems obvious enough, but it is equally obvious that AC and BC are equal to the same thing (AB); yet Euclid makes specific mention of the pertinent axiom here. A complete logical treatment requires that all statements be capable of justification by reference to other statements, officially and explicitly formulated, and not tacitly or unconsciously assumed.

The deficiency can be remedied by inclusion of what is known as a *continuity postulate*, a postulate guaranteeing the existence of certain points under specified conditions. Actually, Euclid's fifth postulate can be interpreted as a kind of continuity postulate, for it asserts that two lines intersect in certain circumstances.

Other examples of logical omissions in Euclid are the following: he assumed without proof, or attempt at definition, that a straight line has two "sides" (as in "interior angles on the same *side* of a transversal"), and that a triangle has an "interior" and "exterior." He used the method of superposition (e.g., placing one triangle on another triangle) in certain proofs without adequate assumptions to justify the method. He neglected to state any postulates concerning such important concepts as the *order* of points in a line, or the concept of one point lying *between* two other points.

Certainly most modern textbooks in high school plane geometry are open to similar criticisms, and often the logical gaps are much more noticeable than in Euclid. In particular, the theory of proportion and areas in the modern texts is rarely as complete logically as in Euclid's treatment.

2–5 Evaluation of Euclid. We now summarize. In the *Elements*, Euclid appears to have produced the world's first treatise embodying the idea of a coherent, logical system of thought. The *Elements* has been called by Sir Thomas Heath the "greatest textbook of elementary mathematics that there has ever been or is likely to be." More than that, it is often classed simply as one of the world's "great books," omitting the textbook designation. Its influence in mathematics, science, education, and philosophy has been tremendous.

Over the course of time, however, certain flaws have become ap-

parent in Euclid's logical system. Among these are the following: (1) there is no official list of technical undefined terms; (2) many concepts requiring definition or clarification by proof are overlooked; (3) some of the proofs of theorems and constructions make use of tacit assumptions not officially formulated as postulates, such as assumptions concerning continuity, order and superposition, and the infinite length of a straight line.

The defects noted can hardly detract from the great value of Euclid's pioneering accomplishment. Certainly comparable defects could be found in the history of art, literature, music, and philosophy.

A number of modern mathematicians have attempted to revise the *Elements* in an effort to transform it into what they believe is a completely rigorous *logical system* of geometry. One of the most influential modern treatments was formulated by the late world-famous German mathematician, David Hilbert, in the last decade of the 19th century. Another well-known revision of Euclid's geometry was given early in the present century by the American mathematician, Oswald Veblen. Both Hilbert and Veblen gave due attention to the listing of undefined terms and the complete statement of the postulates needed for proving all theorems. Furthermore, the postulates themselves were carefully constructed from the standpoint of certain refined criteria of postulational (or "axiomatic") methods. (These criteria will be discussed in detail in Part II; see Chapter 8. Hilbert's postulates will be discussed in Chapter 11.)

EXERCISES

1. Find one or more theorems in Euclid, or in a high school plane geometry textbook, which resort to subconscious assumptions or appearances of the figure in the proof.

2. Give definitions for the word *hypotenuse* and as many as possible of the geometric words used directly or indirectly in defining this word, going back until you reach geometric terms which you think should be left undefined.

3. Investigate Veblen's treatment of the foundations of geometry, noting undefined terms, postulates, and theorems of a fundamental character. See O. Veblen, "The Foundations of Geometry," in *Monographs on Topics of Modern Mathematics*, edited by Jacob W. A. Young, Longmans, Green, 1927, pp. 3–51.

In particular, comparisons with Euclid should be helpful. For example, how is Euclid's equilateral triangle construction validated (cf. Veblen's Assumption XIII and Theorem 34)? What does Veblen do about the two

sides of a line, and the *interior* and *exterior* of a triangle? Does he use the method of superposition for congruent triangles?

2–6 Postulational basis of arithmetic and algebra. Many persons have the mistaken but understandable opinion that geometry is the only branch of mathematics which is amenable to systematic development as a logical system. To counteract this point of view we present in this section illustrations of postulational proofs of some standard procedures in arithmetic and algebra. No attempt will be made at this stage to give any kind of a connected logical treatment of these subjects, but a partial program of this nature will play a prominent role in Part II (see Chapter 6).

It is common practice in elementary algebra to lay stress on the axioms of equality, such as "if equals are added to equals the results are equal," or "quantities equal to the same quantity are equal to each other." Unfortunately, it is customary to give only scant attention to certain assumptions of a more fundamental kind than the equality axioms. These other assumptions concern such apparently simple things as order and combining of terms or factors in the operations of addition and multiplication. We shall refer to five postulates by name. In these postulates a, b, c refer to numbers which may or may not be distinct.

There are two sets of postulates, known as *commutative* and *associative* postulates, dealing separately, but in a parallel manner, with addition and multiplication. Often they are referred to as commutative and associative *laws*. This terminology shows the influence of the absolute truth point of view, but is acceptable if "law" is interpreted as "principle" (see the discussion in Section 5–4). Another pair of postulates, concerned with both addition and multiplication, are the *left* and *right distributive postulates*. We shall see that one of the distributive postulates is logically superfluous in that it can be deduced from the commutative multiplication postulate combined with the other distributive postulate.

The postulates are as follows:

1.	$a + b = b + a$	(Commutative postulate for addition.)
2.	$ab = ba$	(Commutative postulate for multiplication.)
3.	$(a + b) + c = a + (b + c)$	(Associative postulate for addition.)

4. $(ab)c = a(bc)$ (Associative postulate for multiplication.)
5. $a(b + c) = ab + ac$ (Left distributive postulate.)
6. $(a + b)c = ac + bc$ (Right distributive postulate.)

Let us consider the application of these postulates in some of the simplest types of arithmetical computation. For this purpose we use the following illustrations, which will be somewhat reminiscent of the suggested proof in Chapter 1 that $2 \times 2 = 4$, and also of our work with number bases. (Note, however, that here we are using uniformly the base ten.)

Illustration 1. $12 + 3 = 15$.

Proof. Since 12 means $10 + 2$, we write

$12 + 3 = (10 + 2) + 3$ by definition,
$= 10 + (2 + 3)$ by the associative postulate for addition,
$= 10 + 5$
$= 15$ by definition.

(This illustrates a convenient method of proof by continued equalities. Thus $12 + 3$ = each of the indicated expressions on the right, and for the indicated reason.)

Illustration 2. $3 \times 12 = 36$.

Proof. $3 \times 12 = 3(10 + 2)$ by definition,
$= 3(10) + 3(2)$ by the left distributive postulate,
$= 3(10) + 6$
$= 36$ by definition.

Illustration 3. $10 + 20 = 30$.

Proof. $10 + 20 = 1(10) + 2(10)$ by definition,
$= (1 + 2)10$ by the right distributive postulate,
$= 3(10)$
$= 30$ by definition.

The commutative, associative, and distributive postulates are just as important in algebra as in arithmetic. Before illustrating their algebraic use we note that by virtue of the associative postulate the parentheses used in adding or multiplying three numbers may be omitted without ambiguity. Thus we often write

$$a + b + c \text{ instead of } (a + b) + c,$$

and, similarly,

$$a + b + c + d \text{ instead of } [(a + b) + c] + d.$$

Furthermore, the left and right distributive postulates lead to the following generalized left and right distributive rules:

$$a(b + c + d) = ab + ac + ad,$$
$$(a + b + c)d = ad + bd + cd,$$

and similarly if more than three terms are involved.

Now consider the following algebraic illustrations of the postulates. It should be easy from the illustrations and exercises to get some preliminary idea of the unifying effect of presenting algebra as a logical system, instead of as the usual loose collection of rules of procedure. Again we mention that the possibilities in this direction will be more thoroughly investigated in Chapter 6.

Illustration 4. Factor $ab + ac + ad$.

$ab + ac + ad = a(b + c + d)$ by the generalized left distributive rule.

Illustration 5. Combine $3xy + 5xy$.

$3xy + 5xy = (3 + 5)xy$, or $8xy$, by the right distributive postulate.

Illustration 6. Multiply $x + y$ by $z + w$.

$(x + y)(z + w) = (x + y)z + (x + y)w$ by the left distributive postulate,

$= (xz + yz) + (xw + yw)$ by the right distributive postulate,

$= xz + yz + xw + yw$ by the associative postulate for addition.

Exercises

1. Give postulational proofs for the following statements:
(a) $23 + 7 = 30$.
(b) $4 \times 19 = 76$.
(c) $34 + 25 = 59$.
(d) $12 \times 13 = 156$.
(e) $6 \times 20 = 120$.

2. Prove the right distributive postulate as a theorem by use of certain other postulates.

3. Prove the formula for $(a + b)^2$.

4. Multiply the following, justifying each step by postulates:
(a) $(x + 4)(x + 3)$.
(b) $(x + a)(x + b)$.

5. Combine like terms, or factor, justifying the steps:
(a) $7ab + 3ab + 5ab$.
(b) $2axy + 2ayz + 2azw$.
(c) $4(x + y) + 9(x + y)$.
(d) $ax + by + ay + bx$.

2–7 The value of postulational organization in pure and applied mathematics and in science. We now attempt to reach a general estimate of the value of logical systems in both pure and applied mathematics, and in the sciences. We shall often refer to the type of organization used in logical systems as *postulational* organization. By *pure mathematics* we have in mind, roughly, mathematics developed on an abstract, self-contained basis without particular regard to any possible kind of practical applications that may ensue. By *applied mathematics* we mean a body of mathematical subject matter which is definitely related to, or suggested by, some tangible situation — often, though not always, intended for practical use — for example, in physics, astronomy, engineering, business, finance, or statistics.

Before proceeding further it is worth emphasizing that many pure mathematical theories developed for their own sake at one period of mathematical history have later been found to have significant practical applications not anticipated at the time of the original abstract investigations. For example, the theory of complex numbers and functions of complex variables, which originated from the standpoint of pure mathematics in the 18th and early 19th centuries, is today indispensable for electricity, radio, and related fields of physics and engineering. Often the very abstractness of a pure mathematical theory is responsible for its wide range of applications.

On the other hand, there are many cases of useful applications in the reverse direction, that is, from the practical to the abstract. Thus Archimedes used physical and geometric experiments as a basis for contributions in pure geometry. The subject of trigonometry was started by two Greek astronomers, Ptolemy and Hipparchus, for the purpose of assisting in their computations. Similarly, Newton was led to develop fundamental ideas of the calculus in his investigations of motion and velocity. These ideas in turn led to further theoretical developments, and to applications within both mathematics and other fields. Again, in the early 19th century, Fourier's studies concerning heat had useful applications in the pure mathematical theory of functions of real variables.* Thus there has often been close interaction of pure and applied methematics. Whether it is really more important to apply pure mathematics to applied or applied to pure might be hard to say.

Although our illustrations of postulational organization have been

* See E. T. Bell, *The Development of Mathematics*, McGraw-Hill, 1940, p. 149.

limited so far to pure mathematics (e.g., geometry, arithmetic, and algebra), it is a fact that this plan of organization is equally important in applied mathematics. For example, well-developed logical systems are available which describe, on an idealized or hypothetical level, our knowledge of heat, light, electricity, stresses in engineering structures, and moving bodies. Often we find the words "mathematical theory of" used to designate a logical system in one of these applied subjects. The idealized nature of the concepts is indicated by such terms as "perfect gas," "frictionless wheels," and "motion in a vacuum."

These applied theories, in spite of their abstract foundation, often do a remarkable job of predicting and correlating facts which can be verified within satisfactory limits by direct experimental or observational evidence. More than this, they provide a basis for arriving at new knowledge which might not be acquired by purely empirical methods. But the question of truth in applied mathematical theories, *considered independently of observation and experiment*, is much the same as in pure mathematics. All that is really claimed in either case is that *if* the postulates are true, *then* the theorems are true.

We have seen that it is possible to build up conflicting systems, or theories, of *pure* geometry — the Euclidean and non-Euclidean systems. It was mentioned in Chapter 1 that these geometries can also be considered as competing *applied* mathematical theories of physical space. In the same way, conflicting theories are possible in other parts of applied mathematics.

A useful example that will be developed at some length in Chapter 5, in discussing scientific method and knowledge, is the contrast between the mathematical theories of the universe proposed respectively by Newton and Einstein. The Newtonian system is based on a postulational foundation consisting primarily of Newton's three laws of motion and his law of gravitation, together with Euclidean geometry. The Einsteinian system, usually known as the theory of relativity, is based on postulates which conflict in all three respects — motion, gravitation, and geometry — with Newton's system. Both systems can be developed mathematically as logical systems in their own right, and it is the function of the mathematician to carry through the development in each case with the utmost attention to logical rigor. But the choice between the systems on the basis of their degree of apparent physical truth, their correspondence with the external world, must be left to the scientist rather than to

the mathematician. (Actually we shall find that the weight of scientific evidence at the present time favors Einstein rather than Newton, but nevertheless Newton's system continues to be used successfully for most astronomical purposes.)

This brings us to the general role of logical systems within the area of knowledge and research designated somewhat vaguely as "science." We prefer to consider this area as divided into a number of subjects, or sciences, each concerned with some fairly well-delimited range of natural phenomena.

The objective of any one of these sciences is to do more than to gather together observational facts or generalizations of facts within its range of phenomena. Systematization or logical arrangement of such facts and, if possible, causal explanation of them, is always desired. This suggests the need for the same *type* of logical organization as we have found in well-developed branches of pure or applied mathematics. In other words, it is desirable in any science to formulate an official list of fundamental concepts and postulates to be used as a foundation for building theoretically the concepts and principles of the science. The best foundation is one which provides the closest correspondence between theoretical predictions and observed facts. If there is no choice between two competing systems on this basis, that one will probably be preferred which is simpler in its fundamental assumptions and logical structure.

Of course, there are differences in the extent to which the logical structure of a science can be considered analogous to the logical systems of pure and applied mathematics. In some sciences, the plan of organization is more on a systematic descriptive basis than on a mathematical or logical basis. Other sciences, such as physics or astronomy, are inherently of such a nature that applied mathematical theories can be practically incorporated into them, as an integral part of the science. There are obvious advantages in being able to do this, but certainly the history of science shows that we cannot expect finality in knowledge even when this is done. Revision of theoretical organization, as well as revision of factual knowledge in science, seems always to be taking place.

In view of the discussion in this chapter and the preceding chapter, we now venture the following general summary of the value of the postulational organization in any branch of pure or applied mathematics, or in one of the sciences.

(1) Postulational organization singles out a few fundamental con-

cepts and principles which can be considered the key to the logical soundness and structure of the whole subject.

(2) It transforms the subject in question from a loose collection of rules or generalizations into a coherent, logically connected body of statements which can be used for the purpose of deducing new statements, often yielding knowledge which might not be obtained by other methods.

(3) It emphasizes the fact that the truth of certain statements or propositions (the theorems) of the subject can be made to follow from the truth of certain other propositions (the postulates).

(4) It makes possible the study and comparison of conflicting logical theories developed for the purpose of describing or explaining the same natural phenomena.

Exercises

1. Estimate the comparative roles of postulational organization and systematized description in the following sciences: biology, geology, meteorology, astronomy, physics, chemistry.

2. Investigate whether any of the above sciences are in the process of becoming more mathematical and less descriptive.

3. The British philosopher, Viscount Samuel, has attacked Einstein's theory of relativity because of its extreme mathematical abstraction. At a meeting of the British Association for the Advancement of Science on August 10, 1951, as reported in *The New York Times*, he said: "Mathematics cannot convey broadcasting waves from Paris to London or light and heat from the sun to the earth or let the moon lift the tides of the ocean or keep the stars and planets circling eternally. As concepts they cannot provide us with a medium capable of producing any sort of phenomena in the world of nature." Give your opinion of Viscount Samuel's point of view.

2–8 The value of postulational organization in other fields. Finally, let us ask, is there any value in the postulational plan of organization in fields that would ordinarily be considered nonmathematical and nonscientific?

The answer is that mathematical and scientific subjects do not hold a monopoly on such characteristic features of postulational organization as the listing of official assumptions, the formulation of technical definitions (based in the last analysis on certain key undefined terms), and the use of logical reasoning to arrive at new propositions. This plan of organization is useful for systematic development of ideas and principles in any field of thought — for example,

in economics, law, education, and public policy. Furthermore, the relative truth concept is just as pertinent in these fields as in mathematics and science, and often more sadly overlooked.

One famous historical example of a nonmathematical logical system is found in the field of philosophy. The 17th century philosopher, Spinoza, organized his *Ethics* in almost strictly Euclidean fashion, complete with definitions, axioms, postulates, propositions (i.e., theorems), and corollaries. Unfortunately, Spinoza was under the spell of the absolute truth point of view in geometry and seemed to feel that his theorems were indubitably proved. He even wrote "Q.E.D." * after each proof. Other philosophers, of course, have disputed him; they have objected both to his assumptions and to his reasoning processes. However, Spinoza is generally credited with an ambitious and honest attempt to produce a Euclidean system of ethics.

Returning now to the contemporary scene, a good illustration of the general usefulness of the postulational approach is furnished by considering the nature of formal debates. In any debate, whether between two teams or two individuals, the opposing sides attempt to arrive at opposite conclusions through the use of systematic argumentation. The arguments used must always fall back on certain assumed principles or premises. For clarity, it is customary that both sides list certain assumptions about which they agree, and it is desirable that premises adopted separately by the two sides be brought out in the open. The technical words used in the debate should be understood, if possible, in the same sense by both sides. (Confusion may result if this condition is not met. Or a question of definition, e.g., of "democracy," "justice," "fascism," "communism," "socialism," may turn out to be the crux of the argument.) That side deserves to win which succeeds in presenting the most logically sound and complete case in favor of its conclusion, at the same time offering the best refutation of the arguments of the opposite side.

Formal debates do not come to a person's attention very frequently, but everyone is exposed to numerous arguments and points of view, such as those contained in newspaper editorials, magazine articles, books, and radio speeches. Furthermore, every individual often has to formulate arguments of his own to convince himself or others of a desirable opinion or plan of action. A person who is thoroughly familiar with the abstract pattern of a logical system

* *Quod erat demonstrandum*, i.e., "which was to be proved."

can be expected to have an advantage in recognizing sound and unsound logical positions and in constructing valid systems for himself. At the same time he should be more open-minded and tolerant if he appreciates the possibility of arriving logically at positions conflicting with those he favors.

It is a peculiarity of attempts to reach conclusions in fields dealing with social, ethical, political, or international problems, as contrasted with mathematical and scientific subject matter, that so much disagreement is both possible and probable in the conclusion. Of course, this is to be expected by the relative truth principle when conflicting assumptions are adopted in the beginning. But even when the explicit postulates are the same, the conclusions reached by different persons are often varied. It is not hard to account for this, when we consider human nature and all the possibilities for arguments based on concealed or unconscious assumptions involving likes and dislikes, emotions, and ingrained prejudices. This may offer at least a partial explanation of the oft-mentioned lag of social progress behind scientific and technical progress.

As an aid in doing the exercises below concerning logical, or postulational, organization in nonmathematical fields, and also as a reemphasis of some of the foregoing points, the following quotation should be useful. It is taken from a small but eloquent book, *Thinking About Thinking*,* by the late Professor C. J. Keyser of Columbia University, a well-known mathematical philosopher:

> Go to a great library. There you will find many bundles of many kinds; a large bundle of astronomical propositions, a physical bundle, various psychological bundles, communistic bundles, bundles made by advocates of democracy, economic bundles, chemical bundles, ethical bundles . . . and many others. Pick out one of them, examine it carefully, and then let me ask you a question or two. Are the propositions self-evident? If you have half the sense I credit you with, you will say "No." Have they been strictly deduced from a definite body of premises? "I find," you will say, "more or less argumentation but of logical deduction, rightly so-called, there is little or none." Was the author, expositor or advocate a mystic claiming ability to see the truth directly, without mediation of proof? "I discovered no such claim." Did

* C. J. Keyser, *Thinking About Thinking*, New York, Scripta Mathematica, 1942, pp. 86–87. Quoted by permission.

he signalize the propositions that he took for granted? "No." Was he aware that he was employing postulates? "Here and there he seemed to be somewhat aware of it but in general he was not." If he had discovered his postulates and deduced his propositions from them, would he then have been entitled to assert that the propositions are true? "He would not; he would have been merely entitled to say: my propositions are true if my postulates are true."

EXERCISES

(Use the quotation from C. J. Keyser as a guide in making the analyses and criticisms below.)

1. Analyze and criticize from the standpoint of postulational organization the following extracts from the Declaration of Independence. (Consider postulates, definitions, undefined terms, and theorems. Also consider the argument both from the point of view of the Colonies and of the King's government.)

> "We hold these truths to be self-evident, that all men are created equal, that they are endowed by their Creator with certain unalienable Rights, that among these are Life, Liberty, and the pursuit of Happiness. That to secure these rights, Governments are instituted among Men, deriving their just powers from the consent of the governed. That whenever any Form of Government becomes destructive of these ends, it is the Right of the People to alter or to abolish it . . . When a long train of abuses and usurpations . . . evidence a design to reduce them under absolute Despotism, it is their right, it is their duty, to throw off such Government and to provide new Guards for their future security. Such has been the patient sufferance of these Colonies; and such is now the necessity which constrains them to alter their former Systems of Government. The history of the present King of Great Britain is a history of repeated injuries and usurpations all having in direct object the establishment of an absolute Tyranny over these States. To prove this, let Facts be submitted to a candid world. . . ."

(Then follows a long list of "injuries.")

> "We, therefore . . . do . . . solemnly publish and declare, That these United Colonies are, and of Right ought to be, Free and Independent States . . ."

2. Analyze and criticize an argument contained in an editorial, magazine article, or speech, from the standpoint of postulational organization. (Substitute a debate or a legal decision, e.g., of the United States Supreme Court, including the arguments of both sides, if you wish.)

3. Investigate and criticize the postulational organization of Spinoza's *Ethics* (see the reference to Spinoza's *Philosophy* in the Bibliography).

CHAPTER 3

ESSENTIALS OF LOGICAL REASONING, I:

FUNDAMENTAL CONCEPTS, IMPLICATION, RULES OF INFERENCE

3–1 Introduction. The *Principia Mathematica* of Whitehead and Russell. So far our concern has been with the larger organizational aspects of logical systems, especially their postulational foundations, the relations of postulates to theorems or deduced propositions, and the relative nature of the truth of these propositions. A satisfactory study of thinking in any field, however, requires a detailed consideration of the reasoning process itself. Accordingly, the present chapter and the following one will deal with this process, which is fundamental in the construction or critical evaluation of any logical system of thought.

First of all, "logical reasoning," as we use the term, will be understood roughly to be the process used in deducing new statements or propositions from one or more given statements. We shall use "logical inference" or "deductive inference" in practically the same sense as this. The latter terms are slightly more technical and are better adapted to precise definition than "logical reasoning." (Note that we are concerned here with logic in the *deductive* sense although it is possible to study logic in the *inductive* sense also. In the latter sense we often find references to "statistical inference" or "experimental inference.")

The following formal definition is paraphrased, not directly quoted, from a recent textbook on logic:*

> DEFINITION. Whenever one or more statements lead to a new statement which must be accepted whenever the original statements are accepted, *merely by virtue of the form and not the content* of the original statements, the process of obtaining the new statement is called *logical* or *deductive inference.*

This definition stresses, through its emphasis on form, the modern point of view in logic. Much of contemporary logical research

* J. C. Cooley, *A Primer of Formal Logic*, Macmillan, 1942.

belongs to the field known under the various names of "symbolic logic," "formal logic," "mathematical logic." This is the study of principles of reasoning, and numerous related topics, by use of a highly abstract and elaborate symbolism. "Classical logic" (which usually means the study of logic based on the work of the great Greek philosopher and logician, Aristotle) is by contrast a more verbal logic.

We shall follow the modern trend and make use of a certain amount of symbolism. One advantage of doing so is that it becomes easy to concentrate on the form of propositions without being sidetracked by words whose connotations may be misleading or variable. The use of symbolism is conducive to an increase in precision of meaning. At the same time, symbolism puts emphasis on abstractness and generality of treatment, making it convenient to apply the same methods of reasoning in numerous fields. For any person with even a minimum amount of experience with algebraic symbolism, this procedure should make it easier to understand and apply valid methods.

If we were following completely the modern approach to logic, we would attempt the ambitious task of developing logic itself as a formal logical system. This would mean that we would start with a foundation of undefined terms and postulates of logic and proceed to deduce theorems which, like theorems in any branch of mathematics, would be subject to the relative truth principle. Postulates and theorems would be available for use in a double capacity, that is, they would convey substantive information and, at the same time, provide official methods of reasoning for use in later proofs. For definiteness, certain formal rules of procedure probably would be adopted ahead of time; these would be considered outside of the system being developed (just as ordinarily the principles of reasoning used in proving theorems within a branch of mathematics are considered as outside that system).

Undoubtedly, the most famous development of logic as a formal logical system was carried through in the early part of the present century in England by Bertrand Russell and the late A. N. Whitehead in their remarkable three-volume treatise, *Principia Mathematica*. This work has become (perhaps paradoxically) almost a "classical" work in the field of symbolic logic. Not only does it develop formal logic with great exhaustiveness but it leads directly from this into the foundations of pure mathematics. Few mathematicians, however — and certainly not the present writer — would

claim acquaintance with more than a small portion of Volume I of the *Principia Mathematica*.*

In this chapter and the next we shall follow essentially the Whitehead-Russell point of view, but from the standpoint of the *Principia Mathematica* our treatment of logic will be very incomplete and informal indeed. Formalization can best be postponed to a more advanced stage in the study of logic. However, we shall be formal to the extent of stating explicitly certain *fundamental assumptions* and *rules* for deductive reasoning.

3–2 Propositions, general and particular. We shall be especially interested in those statements which can be classified, either actually or hypothetically, as "true" or "false" but not both at the same time. Such statements we shall refer to as *propositions*. We leave the words "true" and "false" undefined, and agree that their meaning is sufficiently understood from a psychological point of view. It is understood that propositions may be investigated, or considered, without committing ourselves as to their truth or falsity.

Most of the propositions which are significant in any chain of logical reasoning are composite propositions. That is, they are made up of subpropositions with the use of such connective words as "if . . . then," "and," "or," "not." These connectives are discussed in the next section.

Some propositions also involve use of such general or indefinite words as "all," "some," "any," "every," and will be referred to as *general* propositions. Propositions which do not involve words of this latter kind will be called *particular* propositions.

In the present chapter we formulate certain rules and principles which hold good for both types of propositions, but we restrict our illustrations almost entirely to the case of particular propositions. In Chapter 4 we introduce some additional rules and concepts needed for an adequate treatment of general propositions. This division should emphasize that deductive logic is not adequately described as reasoning "from the general to the particular."

3–3 Negation; disjunction; conjunction. Fundamental assumptions. There are certain important types of *composite* propositions, obtained by using the words "not," "or," "and," in combination

* A. N. Whitehead and B. Russell, *Principia Mathematica*, second edition, Vol. I, Cambridge University Press, 1925.

with simpler *component* propositions. We proceed to define and illustrate these in turn. From now on we shall use such symbols as p, q, r to stand for propositions. We shall identify p with "p is true" (even if we are only considering p, for it amounts to the same thing to consider the claim that "p is true").

DEFINITION 1. The *negation* of a proposition p is the proposition *not-p*, which is understood to be false when p is true, and true when p is false.

"Not-p" will be symbolized by the use of a curl ($\sim$) *as* $\sim p$. It may be read "p is not the case" or "p is false."

As an example, if p stands for "it is raining," $\sim p$ stands for "it is not raining," or "it is false that it is raining."

DEFINITION 2. The *disjunction* of the two propositions p, q is the proposition p *or* q, or *either* p *or* q, which is understood to be true when *at least one* of the propositions p, q is true, and otherwise false.

"p or q" will be symbolized by use of a wedge ($\vee$) *as* $p \vee q$.

We see that "or" is used here in a *nonexclusive* sense; that is, p or q will still be true when both of the propositions p, q are true. We shall keep to this convention about the nonexclusive use of the word "or" — and likewise the pair of words "either . . . or" — unless otherwise specified.

If we use p again as "it is raining," and q as "the sun is shining," then $p \vee q$ becomes "it is raining or the sun is shining." By our agreement, this is true whenever one or both of the stated things are happening, and otherwise false.

DEFINITION 3. The *conjunction* of p and q is the proposition p *and* q, or *both* p *and* q, which is understood to be true whenever both of the propositions p, q are true, and otherwise false.

"p and q" will be symbolized by use of a dot ($\cdot$), *as* $p \cdot q$, *or often simply* pq, *as for multiplication in ordinary algebra.*

Using the same illustrations for p and q as above, $p \cdot q$ becomes "it is raining *and* the sun is shining."

The symbols introduced for negation, disjunction, and conjunction can be combined to make many different varieties of composite propositions. It will be helpful to think of the wedge ($\vee$) or disjunction symbol as analogous to a plus sign in ordinary algebra, and

the curl ($\sim$), or negation sign, as analogous to a minus sign. The curl, however, must always be attached to a proposition symbol (just as if the notation were p' instead of $\sim p$) and not thought of as an operation connecting two propositions. Parentheses and brackets for grouping will be used when needed for purposes of clarity, as in ordinary algebra. As usual the "multiplication" symbol will take precedence over the "addition" symbol. Consider the following examples:

(a) $p(q \vee r)$: p is true and q *or* r is true.

(b) $pq \vee r$: p *and* q is true or r is true.

(c) $p(\sim q) \vee (\sim p)q$: p *and not-*q is the case or *not-*p *and* q is the case. This is sometimes referred to as the "complete disjunction" of p and q, for it says that one and *only one* of the propositions p, q is true. Thus it corresponds to p *or* q, with "or" used in the *exclusive* sense.

(d) $\sim[(\sim p) \vee (\sim q)]$: it is false that *not-*p *or not-*q is the case.

In these examples, when we say something "is true" or "is false" we are merely translating the composite proposition given in symbolic form. Whether or not the claim is correct depends on what particular propositions p, q, r stand for.

On the other hand, certain composite propositions can be claimed to be *always true* regardless of their particular components. Propositions in this category can be called general *logical principles.*

The preceding discussion and definitions have involved two general logical principles which for definiteness we list as *fundamental assumptions* of our formulation of logic. They correspond respectively to two famous "laws" of classical logic, the *law of the excluded middle* and the *law of contradiction.* In the *Principia Mathematica* these occur as theorems, rather than as postulates. Other formal systems of logic, however, might not be in agreement with both of these assumptions.

ASSUMPTION 1. Law of the excluded middle. *For every proposition p, either p is true* or *p is false. Symbolically, the following is asserted to be true:*

$$p \vee (\sim p).$$

ASSUMPTION 2. Law of contradiction. *For every proposition p, it is false that both p* is *true and p is false. Symbolically, the following is asserted to be correct:*

$$\sim[p \cdot (\sim p)].$$

3–4 Implication. Truth tables. One of the most important and familiar connectives for two propositions is given by the pair of words "if . . . then." In the case of a mathematical theorem the "if" part is the *hypothesis* and the "then" part the *conclusion*. The *proof* of the theorem shows that there is a logical connection between hypothesis and conclusion which requires us to accept the latter when we accept the former, on the additional assumption that the postulates (of the branch of mathematics under discussion) are true. On the other hand, "if . . . then" is often used much more casually, without requiring any logical connection between the two component propositions involved; for example, "if you go, then I will go," "if it rains, then it will get cooler," "if he made that statement, then he was wrong."

Instead of "if . . . then" we shall often use the single word "implies," and any composite proposition in one of the forms, "if p, then q," "if p is true, then q is true," or "p implies q" will be referred to as an *implication, and will be symbolized by use of an arrow, as follows:*

$$p \rightarrow q.$$

(Very often, as in the *Principia Mathematica*, "p implies q" is indicated by the use of a "horsehoe" symbol, as $p \supset q$.)

It is desirable to go somewhat more deeply into the concept of implication and define it in terms of connectives already introduced. For this purpose consider the following proposition, supposedly referring to a particular set of circumstances:

If the weather is favorable, *then* the game will be played.

There are two possibilities corresponding to the "if" clause here, namely, the weather may be favorable or not favorable. In the first case, assuming the proposition to be true, the game will be played. In the second case, we are not told what will happen; conceivably the game may be played in spite of bad weather. Thus we see that at least one of the following must be true (and perhaps both are true): (1) the game will be played; (2) the weather is not favorable. Accordingly, the given proposition seems to amount to the same thing as the following disjunction:

Either the weather is *not* favorable, *or* the game will be played.

To make sure that this says the same thing as before we can retrace steps. Given the proposition in the disjunction form just above, if we make the hypothesis that the weather is favorable, this rules out

the first possibility specified (that the weather is not favorable) and leaves the same conclusion as before, namely, that the game will be played.

This illustration leads us to the following official definition for the abstract implication "p implies q."

DEFINITION 4. The implication p *implies* q (or *if* p, *then* q) means *not-p or q:* that is, p *is false or* q *is true.*

Symbolically, we are saying that $p \rightarrow q$ *is definitionally equal to, i.e., interchangeable with,* $(\sim p) \vee q$. (With our understanding about the use of the curl as a "prime," the latter could also be written simply as $\sim p \vee q$.)

According to our definition, there are two principles of implication which are sometimes considered paradoxical: (a) *a false proposition implies any proposition;* (b) *a true proposition is implied by any proposition.*

To illustrate (a), we are correct in claiming that the following is true:

If [p] 1927 was a leap year, then [q] Lindbergh flew to the moon in 1927.

Here, since the p proposition is false, we know immediately that $p \rightarrow q$ holds in spite of the falsity of q. Of course, the implication would be equally correct if the q proposition were to be modified in accordance with the true facts. However, the falsity of p is more striking if q is obviously false. This agrees with common practice. If a person tells a friend a tall tale, the friend may convey his disbelief by replying, "If that is so, then I'm a monkey's uncle."

To illustrate (b) it is correct to assert the following:

If [p] Shakespeare wrote the plays attributed to him, then [q] there are seven days in a week.

Here, since q is true we know that $p \rightarrow q$ holds, regardless of whether p is true or false.

These illustrations emphasize the possible lack of any direct relationship between the p and q propositions, which is a natural result of our definition of implication. Be this as it may, the definition has amply justified itself in the development of logic both for formal and everyday use.

As an additional aid in understanding how the truth or falsity of composite propositions depends on the truth or falsity of the components of such propositions, the method of *truth tables* is useful.

For example, let us form four columns headed p, q, $\sim p$, $p \rightarrow q$. In the first two columns we indicate all of the possibilities as to truth or falsity of p and q (by use of the symbols T and F, respectively). In the third column we indicate the corresponding "truth value" (T or F) for $\sim p$, and in the fourth column for $p \rightarrow q$, which is the same as $(\sim p) \vee q$. It is apparent that a T will appear in the fourth column whenever the corresponding entries in the second and third columns contain at least one T, and an F otherwise.

p	q	$\sim p$	$p \rightarrow q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

The truth table shows clearly that $p \rightarrow q$ is true for every combination of truth values of p and q except when p is true and q is false.

EXERCISES

(Interpret a double negative statement as a positive statement.)

1. Restate the following in the form p *implies* q, indicating the propositions p and q in each case.

(a) If today is Thursday, then tomorrow is Friday.

(b) If it stops raining, the sun will come out.

(c) Either the instructor is not fair, or Jones will pass the course.

(d) If the ABC store does not refund the money, I will not buy there any more.

2. Restate the following implications, $p \rightarrow q$, in the equivalent form, *not-p or q:*

(a) If he fails to cooperate with the boss, he will lose the job.

(b) If the work is not finished on time, then my name is "mud."

(c) If triangle ABC is isosceles, then base angles A and B are equal.

(d) If the train is late, we shall be late for the appointment.

(e) "If you caught a twenty-pound fish, I will eat my hat."

(f) "If I am not badly mistaken, the party is going to carry out its duty to the country."

3. State the following general principles of reasoning symbolically:

(a) If it is false that p *or* q is true, then p is false and q is false.

(b) If p is false and q is true, then not both p and q are true.

(c) If p implies not-q, then p is false or q is false.

(d) The principle of *reductio ad absurdum:* "if a proposition p implies its own falsity, then p is false."

4. Translate the following into words, and identify any statements which assert general principles of reasoning:

(a) $p \rightarrow (q \vee r)$.
(b) $(p \rightarrow q) \vee r$.
(c) $(\sim p) \vee qr$.
(d) $[p(\sim q) \vee q(\sim p)] \rightarrow (p \vee q)$.
(e) $pq \rightarrow \sim[(\sim p) \vee (\sim q)]$.
(f) $\sim[(\sim p) \cdot (\sim q)] \rightarrow (p \vee q)$.

5. Make a truth table showing truth values of $p \vee q$, pq, $p(\sim q)$, $\sim(p \rightarrow q)$, corresponding to truth values of p and q.

6. A commuter waiting for a local train was pleasantly surprised when an express train unexpectedly stopped at the station. As he jumped aboard the conductor rushed out and said, "This train doesn't stop here, so you can't get on." The commuter replied, "If the train doesn't stop here, then I'm not on it," and walked inside. Was the commuter's contention sound?

7. A radio commercial says: "These sleeping pills are guaranteed to give you a good night's sleep, or your money will be refunded." What agreement seems to be made here regarding a possible refund?

3–5 Deductive inference. The fundamental rule and the chain rule. It is natural to inquire now about the relationship between *implication*, as analyzed in the preceding section, and *deductive inference* as defined in Section 3–1. To understand the respective roles of the two concepts, let us consider again the illustration used in the preceding section just before Definition 4. Let us refer to that implication as proposition (A), and to this proposition adjoin a new proposition (B) so that we now have:

(A) If the weather is favorable, then the game will be played.
(B) The weather is favorable.

From (A) and (B) together we obtain the definite conclusion:

(C) The game will be played.

In other words, if we accept (A) and (B) as true propositions — or *premises* — we *infer* or *deduce* the truth of proposition (C). This is more than we could obtain from (A) alone, as we recall that (A) simply yielded a disjunction of which (C) was one component (the other component being "the weather is unfavorable").

It is important to note that in agreement with the definition of logical inference, the process of arriving at (C) depends only on the form of the premises (A) and (B). Actually (A) is an implication of

the form, $p \rightarrow q$, and (B) says that p is true. The conclusion (C) then asserts that q is true. We see that as far as the reasoning process is concerned, the subject matter under discussion is irrelevant. In fact, any other premises of the same form as (A) and (B) would jointly lead to a conclusion of the same form as (C).

Symbolically we have the following pattern, where we use the familiar symbol, $\therefore$, to stand for "hence" or "therefore":

$$\begin{array}{l} p \rightarrow q \\ \underline{p\qquad\quad} \\ \therefore\ q \end{array}$$

In this scheme the line separating the first two propositions from the third indicates that the process of inference leads us from the assertion of the first two propositions as premises to the assertion of the third. In words, the scheme might be expressed by the use of "since . . . hence," as follows:

Since $p \rightarrow q$ is true and p is true, hence q is true.

Of course, the order of the premises $p \rightarrow q$ and p is immaterial, so that the inference pattern might be changed to the form:

$$\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore\ q \end{array}$$

The pattern of reasoning just described is of such great importance that we shall accept it as a fundamental procedure rule of logic and designate it as the *fundamental rule of inference.* This can be stated as follows:

RULE 1. Fundamental rule of inference. *Whenever an implication, $p \rightarrow q$, is accepted as true, and also the hypothesis (or "if" clause), p, of the same implication is accepted, in these circumstances we must also accept the conclusion (or "then" clause), q, of the implication.*

The distinction between implication and inference is worth re-emphasizing. It is essentially this: the truth of the implication $p \rightarrow q$ does not guarantee the truth of either p or q. But the truth of *both* p and $p \rightarrow q$ does guarantee the truth of q.

It is understood that in place of the propositions p, q in the fundamental rule we can substitute composite propositions of any degree of complexity. For example, the fundamental rule would permit us to make the following inference, where as usual we assume the propo-

sitions above the line as premises (for certain supposed meanings of p, q, r, etc.):

$$\begin{array}{l} (p \vee qr) \rightarrow s[t \vee (\sim u)] \\ \underline{p \vee qr} \\ \therefore\ s[t \vee (\sim u)] \end{array}$$

The fundamental rule of inference is certainly needed in proofs throughout the high school plane geometry course, but in spite of the emphasis on supposedly formal arrangement of proofs in that course, it is probable that the rule receives scant, if any, explicit attention there.

The following line of argument is typical in geometry:

> Since [p] two sides and the included angle of triangle ABC are equal respectively to two sides and the included angle of triangle $A'B'C'$, hence [q] triangle ABC is congruent to triangle $A'B'C'$.

The argument as thus stated is abbreviated. Using p and q as indicated in brackets, the complete argument would appear as follows:

> Since p is true and $p \rightarrow q$ is true (by a special case of a previous general theorem), hence q is true (by the fundamental rule of inference).

Another rule of inference, closely related to the fundamental rule, we shall call the *chain rule.* (It is essentially what is called the "syllogism principle" in the *Principia Mathematica.*) It can be stated as an inference pattern as follows:

RULE 2. Chain rule of inference.

$$\begin{array}{l} p \rightarrow q \\ \underline{q \rightarrow r} \\ \therefore\ p \rightarrow r \end{array}$$

This can be justified without trouble by the fundamental rule, by first assuming p as a hypothesis. This, combined with the first premise above, $p \rightarrow q$, yields q, which in turn combined with $q \rightarrow r$ yields r. Hence from the first two premises we obtain $p \rightarrow r$.

The chain rule can obviously be extended to a larger number of steps of the same kind, e.g.,

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \underline{r \rightarrow s} \\ \therefore\ p \rightarrow s \end{array}$$

One application of the chain rule is to the so-called "analytic" method of proof which is usually mentioned in plane geometry textbooks. For example, to prove a proposition of the form $p \rightarrow s$, this method attempts to use a chain of implications working backwards from the desired conclusion s. Thus the person formulating the proof is supposed to think "I can prove s if I can prove r; I can prove r if I can prove q. But I can prove q from p." Then the whole proof can be put together by starting over again with $p \rightarrow q$ and working forward by the extended chain rule as above.

EXERCISES

I. Complete the blanks in the following sets of propositions, so that each set is in conformity with the fundamental rule of inference. (In each case the first two propositions are to be assumed as premises, so that no question as to the actual truth or falsity of any of the propositions is to be considered.)

1. (a) If it is ten o'clock in Chicago, then it is eleven o'clock in New York.
   (b) It is ten o'clock in Chicago.
   (c) Hence . . . .
2. (a) If today is Thursday, ten days from now will be Sunday.
   (b) Today is Thursday.
   (c) Hence . . . .
3. (a) If today is Thursday, ten days from now will be Monday.
   (b) Today is Thursday.
   (c) Hence . . . .
4. (a) . . . .
   (b) 1956 is a leap year.
   (c) Hence 1956 is a presidential election year.
5. (a) If triangles ABC and $A'B'C'$ are congruent, then angle A = angle A'.
   (b) Triangles ABC and $A'B'C'$ are congruent.
   (c) Hence . . . .
6. (a) If food rationing is to be adopted, then the government will be dictatorial.
   (b) Food rationing is to be adopted.
   (c) Hence . . . .
7. (a) . . . .
   (b) Price controls are to be adopted.
   (c) Hence the country will be saved from inflation.

II. Complete the blanks in the following sets of propositions so that each set is in conformity with the chain rule.

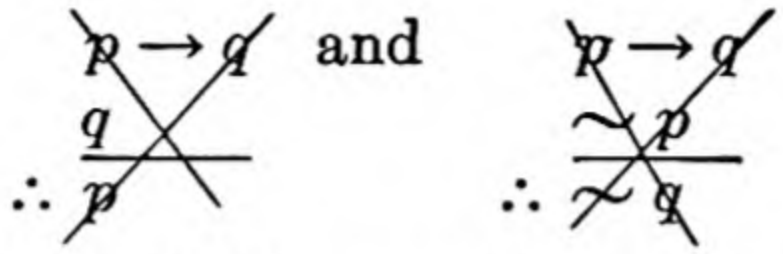
1. (a) *Triangle ABC is equilateral* implies *triangle ABC is equiangular.*
   (b) *Triangle ABC is equiangular* implies *angle A = 60°.*
   (c) Hence . . . .
2. (a) If x is greater than y, then u is less than v.
   (b) If u is less than v, then z is greater than w.
   (c) Hence . . . .
3. (a) If 1900 was a leap year, then 1904 was a leap year.
   (b) . . . .
   (c) If 1900 was a leap year then 1908 was a leap year.
4. (a) If New York time is five hours slower than London time, then Denver time is two hours slower than New York time.
   (b) If Denver time is two hours slower than New York time, then San Francisco time is three hours slower than New York time.
   (c) Hence . . . .
5. (a) Since "X is guilty" implies "Y is innocent," and (b) "Y is innocent" implies "Z is under suspicion," hence (c) . . . .

III. Give illustrations of the use of the fundamental rule and the chain rule in some branch of elementary mathematics.

3–6 The fundamental rule and nonvalid forms. The fundamental rule of inference can be used to deduce a new proposition, q, from premises, $p \rightarrow q$, and p, *regardless of the truth or falsity of the premises.* This should be apparent from the set of exercises at the end of the preceding section. Of course we *assume* the premises to be true as a basis for argument, but this assumption may or may not be correct. Very often we are anxious to find what new proposition follows logically from propositions which we know or believe to be false, or from propositions whose truth is open to question. (We shall see numerous examples of this in connection with our study of such topics as indirect reasoning and scientific method.) In such a case the fundamental rule certainly does not guarantee the truth of the deduced proposition q. But if q is known to be false, the fundamental rule leads us to conclude that either the premise p or the premise $p \rightarrow q$ must be false.

We can summarize by saying that the fundamental rule constitutes a *valid* method of reasoning — a valid argument — irrespective of the actual truth or falsity of premises or conclusions, and it is desirable that this should be so. On the other hand, it is most unfortunate that a *nonvalid* reasoning process may *sometimes* yield true conclusions. For certainly a nonvalid argument *often* yields false conclusions from true premises.

There are two spurious forms of "reasoning" which are closely related to the fundamental rule. These *nonvalid forms* (which we take pains to cross out) are

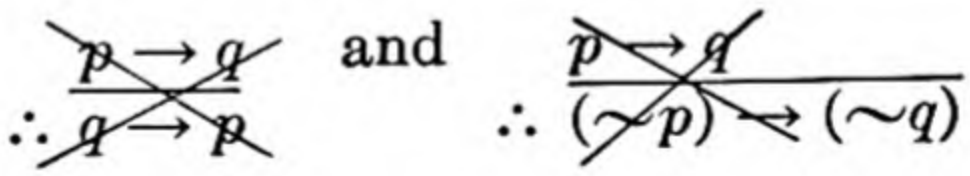


The first of these nonvalid forms claims that if the conclusion (q) of an assumed implication ($p \rightarrow q$) is true, then the hypothesis (p) is also true. The absurdity of this claim is clear from the meaning of $p \rightarrow q$, i.e., "either p is false *or* q is true." We have emphasized previously that the word "or" here is nonexclusive, so that it is perfectly possible, when $p \rightarrow q$ is true, that we have both p false and q true. Thus it is a nonvalid claim in the first form to say that p follows from $p \rightarrow q$ and q. For the same reason it is a nonvalid claim in the second form to say that not-q follows from $p \rightarrow q$ and not-p.

These nonvalid forms are essentially the same as the fallacies of assuming that the "converse" and "opposite" of a given proposition follow from it.

DEFINITION. The *converse* of the implication p *implies* q is q *implies* p. The *opposite* of p *implies* q is *not-*p *implies not-*q.

The arguments just given show that it is possible to have $p \rightarrow q$ true at the same time that we have both p false and q true. In the latter case we certainly have neither the converse proposition ($q \rightarrow p$) nor the opposite proposition ($\sim p \rightarrow \sim q$) true. Thus related *nonvalid forms* are:



EXERCISES

Tell whether or not the following arguments are valid (irrespective of your judgment as to the truth or falsity of the premises or conclusions) and give reasons. If not valid, change the statements so as to obtain an argument which is valid by the fundamental rule of inference. (Note that variations in tense for purposes of clear or grammatical statements are often allowable.)

1. If prices are going still higher, inflation is surely coming.
   Inflation is surely coming.

   Hence prices are going still higher.

2. The days are becoming shorter.
The nights are becoming longer if the days are becoming shorter.

Hence the nights are becoming longer.

3. If today is Tuesday, four days from today will be Saturday.
Today is not Tuesday.

Hence four days from today is not Saturday.

4. If January is hotter than August, then June is colder than November.
January is hotter than August.

Hence June is colder than November.

5. If angle A = angle B, then $AC = BC$.
But we know $AC = BC$.

Hence angle A = angle B.

6. The earth is spherical implies that the moon is spherical.
The earth is not spherical.

Hence the moon is not spherical.

7. If X passes the final examination, then he will pass the course.
X will pass the course.

Hence X will pass the final examination.

8. If Delaware is larger in area than Texas, then Australia is larger than the United States.
Delaware is larger in area than Texas.

Therefore Australia is larger than the United States.

9. If Y gets B or better in all of his courses he will be in good standing.
Y does not get B or better in all of his courses.

Hence Y is not in good standing.

10. The days are becoming shorter.
"The nights are becoming longer" implies that "the days are becoming shorter."

Hence the nights are becoming longer.

3–7 Logical equivalence. Although the converse of a true implication does not *have* to be true, of course, it *may* be true. This leads to the following definition for equivalent propositions.

DEFINITION. If two propositions p, q mutually imply each other, they are said to be *logically equivalent.*

Symbolically, we designate "p is equivalent to q" by a double arrow:

$$p \longleftrightarrow q.$$

It is apparent that two equivalent propositions agree with each other in truth or falsity. For since each of them implies the other, we cannot have one of them true while the other is false. Conversely, if two propositions agree with each other in truth or falsity, we know (by definition of implication) that they mutually imply each other, and hence are equivalent. We see that equivalence of propositions is very much like mathematical equality; in particular, propositions equivalent to the same proposition are equivalent to each other.

Certain forms of composite propositions can be claimed to be always equivalent to each other, regardless of their particular component propositions. Such equivalences, once again, constitute general logical principles. One of the most important of these is the double negation principle, which asserts roughly that a double negative proposition is equivalent to the corresponding positive proposition. We formulate this as an official equivalence rule * as follows:

RULE 3. *For every proposition* p,

$$[\sim(\sim p)] \longleftrightarrow p.$$

This rule can be justified readily by the truth table method introduced in Section 3–4. Thus we have the following (by relying on the law of excluded middle and the law of contradiction):

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

The table shows that p and $\sim(\sim p)$ always agree in truth or falsity — for brevity we can say they agree in *truth value* — and hence must be equivalent. Of course propositions which are "definitionally equal" will be logically equivalent also. For example, we can now write

$$[(\sim p) \vee q] \longleftrightarrow (p \rightarrow q).$$

The value of using equivalences will be seen from the following rule of procedure, which continues the analogy with equality.

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* In a more formalized development the designation "rule" would probably be reserved for a valid method of procedure in deduction, as in the case of our fundamental rule or our rule of substitution, to be introduced as Rule 4. In our development, however, we use the word in a more general sense to cover both procedural methods and general principles of reasoning.

RULE 4. Substitution rule. *Given a composite proposition P, built up by use of the connectives $\vee$, $\cdot$, $\sim$, $\rightarrow$ from a finite number of component propositions p, q, r, . . . , we can substitute for p, q, r, . . . other propositions to which these are respectively equivalent without changing the truth value of P.*

This rule can be justified by noting that the truth value of P is uniquely determined by the truth value of the component propositions, as can be seen by the truth table method. Hence any proposition having the same truth value as one of the component propositions can be substituted for it without any effect on the truth value of P.

3–8 Opposite converse rules. The following definition gives a useful combination of the concepts of "opposite" and "converse" propositions.

DEFINITION. The *opposite converse* (also called *contrapositive*) of the proposition $p \rightarrow q$ is the proposition $(\sim q) \rightarrow (\sim p)$, or more simply, $\sim q \rightarrow \sim p$.

From the definition we see that the opposite converse of a given implication is obtained by interchanging hypothesis and conclusion and inserting "not" in front of each. It is apparent that the opposite converse is at once the *opposite of the converse* and the *converse of the opposite* of the given implication.

We know that neither the converse nor the opposite of a given implication follows logically from it. What about the opposite converse?

Let us assume that the implication $p \rightarrow q$ is true. If we also assume that q is false, then we cannot have p true, for in that case q would be true by the fundamental rule. This would be impossible by the law of contradiction. Thus we have the conclusion that p is false. This can be formulated as a rule of inference, which we shall call the *opposite converse rule:*

RULE 5.* Opposite converse rule of inference.

$$\begin{array}{l} p \rightarrow q \\ \underline{\sim q} \\ \therefore \sim p \end{array}$$

* In any formal development of logic, such as *Principia Mathematica*, this rule would probably appear as a theorem in the form of an implication, asserted to be true for all propositions p, q; namely, (*Footnote cont.*)

The same considerations show that whenever $p \rightarrow q$ is a premise we can obtain from it a new implication

$$(\sim q) \rightarrow (\sim p).$$

In other words, the opposite converse of a given implication follows logically from it. Accordingly, the opposite converse of

$$(\sim q) \rightarrow (\sim p)$$

follows from it also. This would give us

$$\sim(\sim p) \rightarrow \sim(\sim q).$$

But by the double negation rule (Rule 3) we know that

$$\sim(\sim p) \longleftrightarrow p \quad \text{and} \quad \sim(\sim q) \longleftrightarrow q.$$

Hence we can make substitutions according to Rule 4 and obtain

$$p \rightarrow q.$$

Any proposition which *follows logically* (i.e., can be inferred) from a given proposition is surely *implied* by that proposition (although we know that the converse of this statement does not hold). Accordingly, we can state as a general logical principle that an implication and its opposite converse are logically equivalent. This we formulate as the *opposite converse equivalence principle:*

RULE 6. Opposite converse equivalence rule.

$$(p \rightarrow q) \longleftrightarrow (\sim q \rightarrow \sim p).$$

We could formulate variations of the opposite converse rule and the opposite converse equivalence to provide for negation signs in dif-

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$$[(p \rightarrow q) \cdot (\sim q)] \rightarrow \sim p.$$

This implication by itself is not strictly the same as Rule 5, but taken in combination with the fundamental rule it would easily justify Rule 5. For, assuming the premises

$$p \rightarrow q,$$

and

$$\sim q,$$

we have the conjunction

$$(p \rightarrow q) \cdot (\sim q).$$

Then writing the theorem,

$$[(p \rightarrow q) \cdot (\sim q)] \rightarrow \sim p,$$

we obtain by the fundamental rule the conclusion,

$$\sim p.$$

ferent positions, but it is really unnecessary to do this, as the essential principle would remain the same. For example, if we replace p by $\sim p$ in Rule 5, we obtain

$$\begin{array}{l} \sim p \rightarrow q \\ \underline{\sim q} \\ \therefore \sim(\sim p) \end{array} \qquad \text{that is,} \qquad \begin{array}{l} \sim p \rightarrow q \\ \underline{\sim q} \\ \therefore p \end{array}$$ by Rule 3.

The following are some illustrations of the use of the opposite converse methods.

Illustration 1. The following set of statements conforms with Rule 5.

If triangle ABC is isosceles, then angle A = angle B.
But angle A does not = angle B.

Hence triangle ABC is not isosceles.

Illustration 2. In a college catalog we read the following: "If a student is to graduate *summa cum laude*, he must have a cumulative average of 94 or above." Assuming that the stated general requirement applies in the case of a specified student, Jones, we obtain by Rule 6:

If Jones does not have a cumulative average of 94 or above, then he is not to graduate *summa cum laude*.

Illustration 3. Suppose we wish to draw a conclusion from the following two premises:

If the investigation is purely political, it is not desirable.
The investigation is desirable.

Using the following abbreviations:

p: the investigation is purely political,
q: the investigation is desirable,

we see that we are given

$$p \rightarrow (\sim q),$$
$$q.$$

Since q is equivalent to $\sim(\sim q)$, we obtain by the opposite converse rule the conclusion

$$\sim p,$$

that is:

The investigation is *not* purely political.

EXERCISES

1. Fill in the blanks in the following statements, by use of one of the opposite converse rules. State symbolically the particular form of the rule which is used in each case.

(a) If C is on the perpendicular bisector of line segment AB, then it is equidistant from points A and B.
Hence if . . . .

(b) If lines AB and CD are parallel, then alternate interior angles x and y are equal.
But angles x and y are not equal.
Hence . . . .

(c) If Brown does not pass the language requirement, he does not graduate.
Hence if . . . .

(d) If it is not raining, the sun will come out.
The sun will not come out.
Hence . . . .

(e) The car will not break down if it has been properly cared for.
The car will break down.
Hence . . . . (Note the changed position of "if.")

2. Decide whether or not the following arguments are valid. Justify your verdict by reference either to a valid rule of inference or to a nonvalid form.

(a) If A is guilty, he must have been at the scene of the crime at the time it was committed.
But he was not at the scene at the time.

Hence A is not guilty.

(b) If the patient has scarlet fever, he must have a temperature above 100°.
The patient's temperature is not above 100°.

Hence the patient does not have scarlet fever.

(c) If B traveled one mile in 75 seconds, he was exceeding the speed limit.
B did not travel one mile in 75 seconds.

Hence B was not exceeding the speed limit.

(d) If war is not inevitable, then universal military training (UMT) is a necessity.
UMT is a necessity.

Hence war is not inevitable.

(e) If UMT is a necessity, then war is inevitable.
War is not inevitable.

Hence UMT is not a necessity.

3–9 Necessary and sufficient conditions. In mathematics it is common to state theorems in forms like the following: "a necessary condition that p be true is that q be true," or "a sufficient condition that p be true is that q be true." It is worth while to consider this terminology in detail, especially as it has possibilities of wide application.

DEFINITION 1. The proposition, *a necessary condition that p be true is that q be true* (or *q is a necessary condition for p*), means *p implies q* ($p \rightarrow q$).

DEFINITION 2. The proposition, *a sufficient condition that p be true is that q be true* (or *q is a sufficient condition for p*), means *q implies p* ($q \rightarrow p$).

DEFINITION 3. The proposition, *a necessary and sufficient condition that p be true is that q be true*, means *p and q mutually imply each other* (i.e., p and q are *logically equivalent*, $p \longleftrightarrow q$).

Since the implications $p \rightarrow q$ and $q \rightarrow p$ in Definitions 1 and 2 are converses of each other, we see that there is no guarantee that a necessary condition will also be a sufficient condition, or that a sufficient condition will also be a necessary one. Consider the following illustrations from Euclidean geometry.

Illustration 1. A *necessary* condition that a given quadrilateral $ABCD$ be a rectangle is that it be a parallelogram. This is obviously not sufficient (for $ABCD$ can be a parallelogram without being a rectangle).

A *sufficient* condition that $ABCD$ be a rectangle is that it be a square. This is obviously not necessary (for $ABCD$ can be a rectangle without being a square).

A *necessary and sufficient* condition that $ABCD$ be a rectangle (in Euclidean geometry) is that it be a parallelogram with one angle a right angle.

Illustration 2. A *necessary* condition that a given integer N be exactly divisible by 9 is that it be divisible by 3. This is far from sufficient; consider such integers as 6, 12, 15.

A *sufficient* condition that N be divisible by 9 is that it be divisible by 18. This is surely not necessary, for any odd multiple of 9 fails to be divisible by 18.

A *necessary and sufficient* condition that N be divisible by 9 is that the sum of its digits be a multiple of 9.

One more item of terminology is important — the use of "if and only if" as an alternative way of indicating logically equivalent propositions, or necessary *and* sufficient conditions. It is sufficient here to define "only if."

DEFINITION 4. The proposition, *p is true only if q is true*, means *q is a necessary condition for p* (i.e., *p implies q*).

This often causes trouble, since the position of "if" is misleading. The emphasis should be on "only," thus: "the *only* way in which p can be true is to have q true also," which clearly asserts that p implies q, or that q is a *necessary* condition for p.

On the other hand, "p is true *if* q is true" says that q is a *sufficient* condition for p. To prove that "p is true *if and only if* q is true," we should prove the "if" part and the "only if" part separately.

A mathematical or logical definition can be interpreted as a necessary and sufficient condition, but usually the form of the definition does not indicate this. For example, when we say "triangle ABC is defined to be equilateral *if* sides AB, BC, CA are equal," we surely understand "if and only if" in place of "if."

EXERCISES

1. Rephrase the following quotations in the most natural way as necessary or sufficient conditions. (All were taken from the *New York Times* for August 10, 1947.)

(a) "If the Western world is to achieve economic stabilization . . . it will have to put economic recovery above all other considerations."

(b) "If the plans developed for a permanent headquarters of the United Nations are carried out on the scale outlined, the world organization will have a truly impressive workshop."

(c) "Holland's experts . . . are convinced that if the rubber, petroleum, sugar, and coffee trades can once more get underway, Holland's financial situation and her need for dollars will be ameliorated."

(d) "To control hay fever the proper allergens must be administered in a series of doses."

(e) "The entire Alaska Highway will be closed if the huge Peace River suspension bridge is in danger of collapsing."

(f) "Hitherto the sulfa drugs have been successful only if the patient could be treated in less than ten days after contracting the disease undulant fever."

(g) "The Democrats . . . will have to take up the cudgels of the New Deal if they would successfully defend the liberal tradition."

2. The following quotation is taken from a speech by Mr. Clement Atlee of Great Britain, November 9, 1946 (at which time he was Prime Minister):

> "If the U.N. organization is to be used as a forum for debating ideological differences it will not succeed. It can only succeed if it is to be used to secure to all nations the freedom to preserve their own ways of life while contributing to the common good of the world."

What kind of condition, or conditions, did Mr. Atlee propose for the success of the U.N. — necessary, sufficient, or necessary *and* sufficient?

3. (a) Restate the following as a necessary or sufficient condition:

> "It is clear that elimination of standee passengers can be brought about only if additional heavy expenditures are made on the part of the Long Island Railroad."

(From a brief filed by the Long Island Railroad with the New York State Public Service Commission, November 13, 1946.)

(b) The next sentence following the one just quoted reads:

> "It is equally clear that the Long Island is not financially able to make such expenditures."

Do the two sentences, taken jointly, strictly allow us to reach any conclusion about "elimination of standee passengers"? If not, what additional premise would permit us to reach a conclusion?

4. (a) Formulate three theorems in Euclidean plane geometry by use of "if and only if" and identify (by stating "given" and "to prove") the two parts of each theorem.

(b) Change each theorem to the "necessary *and* sufficient condition" terminology, and identify the two parts again.

5. (a) How is the word "provided" used in ordinary conversation (as in "I will go provided you go also")? Does it in your opinion express a *necessary* condition, *sufficient* condition, or *necessary and sufficient* condition? Or does it vary with circumstances?

(b) The same for the word "unless."

6. Given: (a) p is true if and only if q is false; (b) q is a sufficient condition for the falsity of r; (c) r is true; (d) s is a necessary condition for p.

Obtain a conclusion, using all of these premises and showing in detail what rules of inference are being used.

7. Obtain a conclusion from the following premises:

If the investigation will result in mudslinging, it is purely political.
The investigation is desirable only if it is not purely political.
The investigation is desirable.

3–10 Rules governing disjunction and conjunction. So far, the rules of inference we have listed have not explicitly involved disjunc-

tion or conjunction. In this section we introduce a few useful rules involving these logical operations.

We start by noting that the truth of the conjunction, pq, is a sufficient condition, but not a necessary condition, for the truth of the disjunction $p \vee q$. Hence we cannot argue from the premises, $p \vee q$ and q, either that p is true or p is false. However, if $p \vee q$ holds and one of the propositions p, q is false, it is surely necessary that the other one be true. This principle gives us the following double rule of inference:

RULE 7. (a) $\dfrac{p \vee q \qquad \sim p}{\therefore q}$ (b) $\dfrac{p \vee q \qquad \sim q}{\therefore p}$

Incidentally, we note that

$$(p \vee q) \longleftrightarrow [\sim(\sim p) \vee q]$$

by Rules 3 and 4 and

$$[\sim(\sim p) \vee q] \longleftrightarrow [(\sim p) \to q]$$

by the definition of implication, whence

$$(p \vee q) \longleftrightarrow [(\sim p) \to q]$$

by the chain rule. Accordingly, Rules 7(a) and 7(b) are essentially the same respectively as:

(a) $\dfrac{(\sim p) \to q \qquad \sim p}{\therefore q}$ and (b) $\dfrac{(\sim p) \to q \qquad \sim q}{\therefore p}$

These two forms are simply variations of the fundamental rule and the opposite converse rule of inference.

Suppose now that $p \vee q$ is false. This can only happen when both p is false and q is false. Conversely, if both p and q are false, then $p \vee q$ is certainly false. Hence we can say that $\sim(p \vee q)$ is equivalent to $(\sim p) \cdot (\sim q)$.

Again, if pq is false it must be the case that either p is false or q is false, and conversely. Thus $\sim(pq)$ is equivalent to $(\sim p) \vee (\sim q)$.

These two equivalences are known as DeMorgan's laws, and are listed here as Rules 8 and 9.

RULE 8. $\sim(p \vee q) \longleftrightarrow (\sim p) \cdot (\sim q)$.
RULE 9. $\sim(pq) \longleftrightarrow (\sim p) \vee (\sim q)$. } *DeMorgan's laws.*

There are certain other equivalences which are of considerable theoretical importance in any systematic development of symbolic logic, or in the theory of Boolean algebras (the abstract version of the "algebra of logic") to be introduced in Chapter 9. For application to reasoning techniques, however, these are not of great importance to us, so we do not list them as formal rules. They can be arranged in parallel columns, showing corresponding properties of disjunction and conjunction:

1. $(p \vee p) \longleftrightarrow p$.	1′. $pp \longleftrightarrow p$.
2. $(p \vee q) \longleftrightarrow (q \vee p)$.	2′. $pq \longleftrightarrow qp$.
3. $[(p \vee q) \vee r] \longleftrightarrow [p \vee (q \vee r)]$.	3′. $(pq)r \longleftrightarrow p(qr)$.
4. $p(q \vee r) \longleftrightarrow (pq \vee pr)$.	4′. $(p \vee qr) \longleftrightarrow (p \vee q)(p \vee r)$.

Note that we can change either column into the other by the simple process of interchanging the symbols, $\vee$ and $\cdot$, for disjunction and conjunction. For this reason the two lists of properties are said to be *duals* of each other, and to exhibit complete *duality*.

In this list, equivalences 2, 2′, 3, 3′, and 4 are analogous to the usual commutative, associative, and distributive postulates of ordinary algebra. Numbers 1 and 1′ and 4′ are unlike any rules of ordinary algebra. All of the equivalences can be verified directly by the definitions of disjunction and conjunction. All except 4 and 4′ are easily checked.

To verify number 4, we see that the left side of the equivalence

$p(q \vee r)$ means: p is true and q *or* r is true.

This in turn means that either

p *and* q or p *and* r is true,

which is precisely the meaning of the right side of the equivalence.

To verify number 4′, we see that the right side

$(p \vee q) \cdot (p \vee r)$ means: p *or* q and p *or* r is true.

This indicates that either p is true or q *and* r is true, which agrees with the meaning of the left side.

We conclude this section with an illustration showing the combined application of various rules or definitions.

Illustration. Obtain one conclusion from the following three premises:

(1) If moderate political leaders can be returned to power and ex-

tremists can be removed, then there will be a balance in the government and the danger of a new dictatorship can be neutralized.

(2) The danger of a new dictatorship cannot be neutralized.

(3) Extremists can be removed.

Notation. Introduce the following symbols:

p: Moderate political leaders can be returned to power.

q: Extremists can be removed.

r: There will be a balance in the government.

s: The danger of a new dictatorship can be neutralized.

Given: $pq \rightarrow rs,\ \sim s,\ q.$

Development.

	$rs \rightarrow s$	by definition of conjunction,
	$\sim s$	by hypothesis,
	$\therefore \sim(rs)$	by opposite converse rule,
	$pq \rightarrow rs$	by hypothesis,
	$\therefore \sim(pq)$	by opposite converse rule.
But	$\sim(pq) \longleftrightarrow (\sim p) \vee (\sim q)$	by one of DeMorgan's laws,
	$\therefore (\sim p) \vee (\sim q)$	by substitution rule (or fundamental rule),
	$\sim(\sim q)$	by hypothesis and double negation rule,
	$\therefore \sim p$	by Rule 7(b).

Conclusion: Moderate political leaders cannot be returned to power.

EXERCISES

1. (a) Show that $(p \rightarrow q) \longleftrightarrow \sim[p \cdot (\sim q)]$ and interpret this equivalence in words.
   (b) State a proposition equivalent to $\sim(p \rightarrow q)$.
2. Are the following arguments valid? Why or why not?
   (a) AB is parallel to EF or CD is parallel to EF.
   AB is parallel to EF.

   Hence CD is not parallel to EF.

   (b) AB is parallel to EF or CD is parallel to EF.
   CD is not parallel to EF.

   Hence AB is parallel to EF.

   (c) Either A is not guilty or B is telling the truth.
   B is not telling the truth.

   Hence A is not guilty.

(d) A is not guilty and B is not telling the truth.

Hence it is not the case that "A is guilty or B is telling the truth."

(e) It is not the case that "A is guilty and B is telling the truth."

A is not guilty.

(f) If Smith is preparing for medicine he is required to take both biology and chemistry.
Smith is not required to take chemistry.

Hence Smith is not preparing for medicine.

3.* *Given* the premises: If A accuses B or C then he is serving D's interests. A is serving D's interests if and only if he cooperates with E and F.
Prove: If A does not cooperate with E he will not accuse B.

4.* (Review) Determine whether the proposed conclusion follows logically from the given statements about a city lighting system:

The generators of the city power plant are operated simultaneously only if the street lights are on.
If the load is over 10,000 kw, then the generators are operated simultaneously.
The street lights are on.

Hence, the load is over 10,000 kw.

5. Change the third premise of the preceding exercise to "the street lights are not on." Obtain a valid conclusion, leaving the other assumed statements as they were.

6. Obtain one conclusion from the following five premises (which are based on an article which concerned the economic situation in Germany soon after the end of World War II). Use p, q, r, etc. to symbolize *positive* statements, and indicate the meaning of the notation explicitly before starting the logical development.

(a) If there is not a sufficient supply of commercial fertilizer, or if there is not an adequate transportation system in Germany, Germany's farms cannot make available sufficient food to maintain the German population.

(b) If there is a shortage of coal in Germany, then there is both a shortage of power and a shortage of steel.

(c) If there is a shortage of power, then there is an insufficient supply of commercial fertilizer.

(d) If there is a shortage of steel then there is not an adequate transportation system.

(e) There is a shortage of coal in Germany.

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* Taken from J. C. Cooley, *A Primer of Formal Logic*. The Macmillan Company, copyright 1942. Used by permission of the publishers.

3–11 Indirect reasoning. Any person who has been exposed to high school geometry will recall that certain theorems of the course were established by the "indirect method of proof." Euclid used the method quite freely. Some high school students object to the indirect method as "beating around the bush." Actually the method is of great importance, not only in mathematics but in everyday affairs and legal and scientific reasoning as well.

We shall distinguish three different varieties of indirect reasoning (primarily for use in proving theorems) which will be designated as Methods I, II, III.

Method I involves proving a proposition of the form $p \rightarrow q$, by proving instead the opposite converse equivalent, $\sim q \rightarrow \sim p$, on the assumption that the underlying postulates are true.

As an example of this method, suppose we are asked to prove the Euclidean theorem, "if two lines are parallel, the alternate interior angles (formed by a transversal of these lines) are equal." Instead, we may prove the opposite converse theorem, "if the alternate interior angles of two lines are not equal, then the lines are not parallel." This is readily accomplished as follows. (We assume here the common sequence of theorems in which the *converse* alternate interior angle theorem has already been proved, i.e., "if the alternate interior angles of two lines are equal then the lines are parallel.")

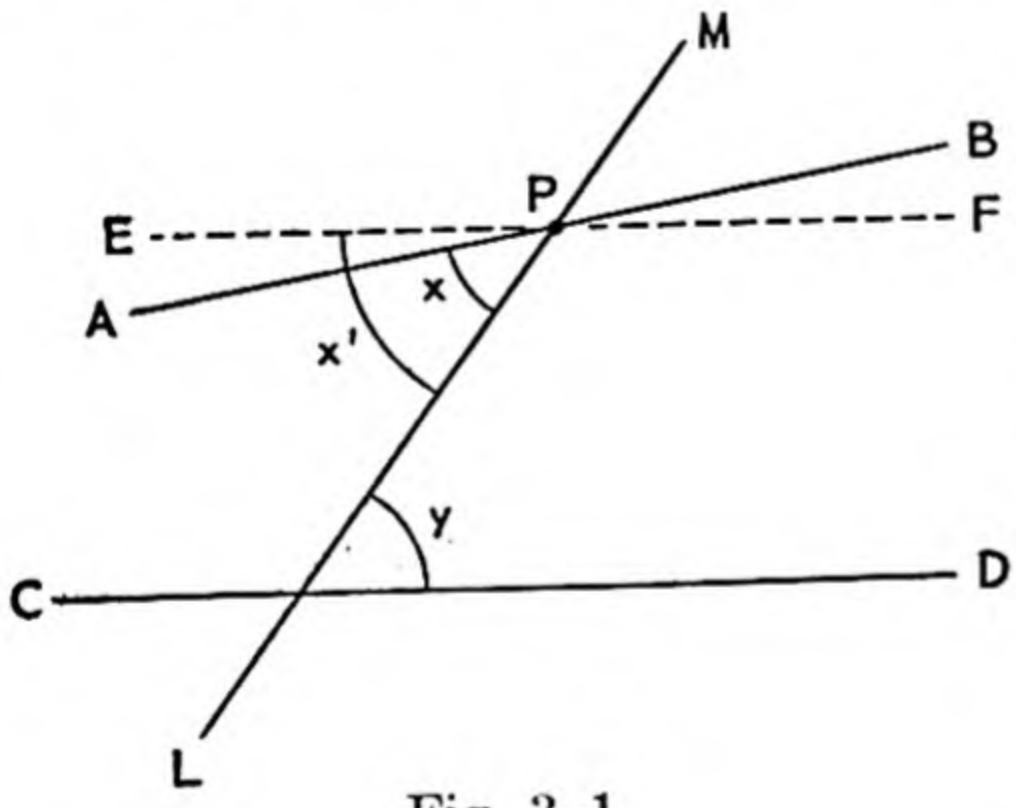

Fig. 3–1

Given: Lines AB, CD cut by transversal LM with angle $x \neq$ angle y.

Prove: AB is not parallel to CD.

Proof. At P, the intersection of AB with transversal LM, construct line EF making

angle x' = angle y.

EF must be distinct from AB. By the converse alternate interior theorem (mentioned above) it follows that

EF is parallel to CD.

By the Euclidean parallel postulate we cannot have more than one line through P parallel to CD. Hence AB is not parallel to CD.

Note that this proof does not require use of a distorted figure, nor statements which appear confusing and contradictory to a beginner. The theorem originally required is now immediately available as a corollary based merely on the opposite converse equivalence principle.

What we shall call *Method II* depends essentially on another logical equivalence, namely,

$$(p \to q) \longleftrightarrow \sim[p \cdot (\sim q)].$$

This equivalence was given in Exercise 1 at the end of the preceding section. It is easily justified by using one of DeMorgan's laws to reduce the proposition on the right of the equivalence sign to the definition of the implication $p \to q$. (Or we can refer back to our earlier statement that $p \to q$ is true in every case except that in which p is true and q is false, so that $p \to q$ and $\sim[p \cdot (\sim q)]$ agree in truth value.)

In order to prove a theorem $p \to q$ by Method II, the procedure then is to show that it is impossible to have the hypothesis true and the conclusion false. Thus the hypothesis is kept intact and the supposition is made that the conclusion is false, after which it is attempted to show that these jointly lead to a contradiction of some kind — a contradiction either of a previous theorem, or of the hypothesis, or of the supposition. In case such a contradiction, say $r \cdot (\sim r)$, materializes, we have the following (implied by the fundamental postulates):

$$p \cdot (\sim q) \to r \cdot (\sim r).$$

But $\sim[r \cdot (\sim r)]$ by the law of contradiction

Hence $\therefore \sim[p(\sim q)]$ by the opposite converse rule.

$p \to q$ by the equivalence mentioned above.

From the standpoint of underlying logical principles it appears that Method II is more complicated than Method I. Also, in our illustration of the alternate interior angle theorem, Method II has the additional disadvantage of causing psychological difficulties, due perhaps to apparent conflicts between the figure and the accompanying reasoning — for example, drawing two lines as separate lines, and then saying they must coincide. In other cases, however, Method II may be more workable than Method I.

Method III is the familiar process of ruling out possibilities. Thus, for two real numbers a and b we may assert that one of three statements is true:

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$$a < b, \quad a = b, \quad \text{or} \quad a > b.$$

Suppose we are able to show that the first two possibilities are false; then the third must be true. From the logical standpoint this is based on an obvious extension of our Rule 7 above, namely,

$$\begin{array}{l} p \vee q \vee r \\ \sim p \\ \sim q \\ \hline \therefore r \end{array}$$

It is possible, of course, that $\sim p$ or $\sim q$ themselves may be established by Method I or Method II.

Finally, let us consider briefly the use of indirect reasoning in fields other than mathematics.

It should be apparent that any argument in which a rule of inference of opposite converse type is used represents an instance of indirect reasoning. Thus whenever a person is defended in a criminal case by claiming an alibi, he is relying on indirect reasoning. The argument of his lawyer may go as follows:

> If my client committed the crime, he must have been at the scene of the crime at the time it was committed. But he was *not* at the scene at the time (for at that time he actually was in another city 100 miles away). Hence he did not commit the crime.

This simple application of the opposite converse rule is typical of *units* of reasoning that enter into a lengthy process of crime detection, involving tracking down and elimination of suspects, and finally pointing (perhaps) to one guilty person. Such a process may make use of all of the Methods I, II, III, and many different patterns of reasoning are conceivable.

EXERCISES

1. Use Method II to prove the following theorem: If two lines are parallel, then the alternate interior angles of these lines are equal.

2. Prove: If two angles of a triangle are unequal, then the sides opposite these angles are unequal, the greater side being opposite the greater angle. (Assume that you have already proved the isosceles triangle theorem; and also the theorem saying that if two sides of a triangle are unequal, then the angles opposite are unequal, the greater angle being opposite the greater side.)

3. Prove: If a given line segment subtends a right angle at a given point, then the point is on a circle having the given segment as diameter.

4. Describe a typical process of reasoning used in attempting to locate a lost object.

5. Find examples from (a) science or medicine, (b) everyday affairs, of the use of indirect reasoning, and give the symbolic formulation of the method.

6. Investigate and analyze Euclid's indirect proof that the number of prime numbers is infinite. (This is a classical illustration of the use of the principle known as *reductio ad absurdum*.) See, for example, R. Courant and H. Robbins, *What is Mathematics?*, Oxford University Press, 1941, p. 22.

3–12 Classification of errors in reasoning. In this final section of the chapter we bring together and classify some common errors of reasoning.

We shall not discuss again the very familiar nonvalid arguments which are based on assuming that the converse and opposite of a given implication follow from it. These fallacies were treated in Section 3–6. Here we shall list them simply as the fallacies of (1) *assuming the converse* and (2) *assuming the opposite*. (Alternative names in common use for these are respectively "affirming the consequent" and "denying the antecedent.")

Other errors that we shall now consider are: (3) the *non sequitur* error, (4) the *post hoc ergo propter hoc* argument, (5) *begging the question* (or *petitio principii*), and (6) *statistical fallacies*. In Chapter 4 some additional fallacies dealing with classes will be introduced.

Non sequitur. In a sense, the fallacies of assuming the converse or opposite, and perhaps all logical errors, are special cases of the general *non sequitur* error. "Non sequitur" means "it does not follow." We shall reserve the name for cases which are not classified conveniently under one of the more specialized headings.

A typical abstract pattern of a *non sequitur* would be "since p is true, therefore q is true." This is like the pattern of the fundamental rule with the premise $p \rightarrow q$ omitted. Of course if this premise is known to be correct, the argument is valid, though abridged. But conceivably the correct premise to assume might be

$$p \rightarrow \sim q,$$

in which case the correct conclusion would be $\sim q$ instead of q.

A possible example of this kind of *non sequitur* would be the following:

> Since [p] the government must protect the American way of life, hence [q] it is justifiable for the government to restrict freedom of speech for political extremists.

This is precisely an argument in which, for the indicated p and q, some people would claim that $p \rightarrow \sim q$ is correct rather than $p \rightarrow q$.

Post hoc ergo propter hoc. This is a special type of *non sequitur*, which literally means "after this, therefore on account of this." Thus it is first asserted that event B came after event A, and then it is claimed or strongly suggested that event B was *caused* by event A. Every voter should be on the alert for arguments of this kind in election campaigns.

Of course it may be that the *post hoc* line of argument can be justified in specific cases. The important thing is, however, that evidence is required in each case on its merits to show that event A caused, or was largely instrumental in causing, event B.

Begging the question (or *petitio principii*). The fallacy here is that in attempting to prove a certain proposition the desired proposition itself is unwittingly assumed in the argument. Thus the argument degenerates into an unconvincing assertion of the type, "it's true because it's true," which has no more force than a straight assumption of the proposition in question. From another standpoint, the argument might be called a "vicious circle" fallacy.

Begging the question is a common error in the work of beginning students in high school plane geometry. For example, in proving that the base angles of an isosceles triangle are equal, a student may assume these angles equal in order to prove two triangles congruent, and then go on to argue that the angles in question must be equal as they are corresponding parts of congruent triangles! In other cases, the question may be begged in a more subtle, although equally pernicious, manner.

Statistical fallacies. Many examples of faulty reasoning occur in arguments of a statistical nature. Often these can be classified under one of the preceding headings, such as statistical *post hoc* or *non sequitur* fallacies. We should be on the alert for fallacies of this kind when we see claims asserting that "statistics prove so and so." Examples of this nature abound in modern advertising.

More detailed discussion of such fallacies can be found in statistics textbooks. The statistical trouble may lie in things like the following: misleading use of means or averages, unjustifiable comparisons of data, faulty use (or lack of use) of per cents.

EXERCISES

1. What procedure is commonly used in scientific investigations (especially in biology, medicine, or psychology) to avoid the *post hoc* fallacy in estimating the extent to which one event is the cause of another event? Illustrate.

2. Decide whether or not the following arguments (or suggested arguments) are valid. If not valid, classify the type of error in the most appropriate way, and find the particular source of the fallacy. (There is often room for some disagreement here.)

(a) "The Democrats cannot blame the Republicans for high prices because in June, 1946, when price controls were removed, the Republicans were not in office." — From a letter in the *New York Times*.

(b) "Some of the recent increases in food prices have been attributed both to short supply and to our exports of food to foreign countries. The truth is prices have risen largely in line with a general upward trend." — From a newspaper editorial.

(c) "Since the yearly average temperature of our city is 70°, it is an ideal all-year-round resort." — From a travel advertisement.

(d) In *Alice in Wonderland* the March Hare has put butter in the works of the Hatter's watch, and the Hatter accuses him of spoiling it. The March Hare defends himself by replying, "it was the *best* butter."

(e) The following is from *Alice in Wonderland* again: "'How do you know I'm mad?' said Alice. 'You must be,' said the Cat, 'or you wouldn't have come here.' Alice didn't think that proved it at all."

(f) During the first year in which a certain graduate school was operated on a coeducational basis, several faculty members were strongly opposed to the plan. At the end of the year they reported that the consequences of this action might well be disastrous, inasmuch as 33⅓% of the women students had married into the faculty.

(g) "I do have one thing, one of the most expensive collections of jewelry in Hollywood. I've been married three times, so you know I came by it honestly." — Remarks by a movie star.

(h) "It is true that the removal of rent ceilings would be followed by an increase in rents, but this would not necessarily lead, in the long run, to an increase in living costs. For with more of consumers' incomes being paid for rent, just that much less would be left to bid up the prices of everything else." — Remarks of a commentator in a weekly periodical.

(i) "The Moscow Evening News quoted Prof. A. B. Vorontsov as having said that it had been discovered that the atmosphere on Mars contained approximately the same percentage of carbon dioxide as that on the earth. This represents reasonable proof of the contention of Soviet Scientist Tikhov, the News quoted the professor as having said, that there is plant life on Mars, since without plant life the carbon dioxide content would be much higher." — From an Associated Press dispatch.

(j) "That unusual hereditary endowment is necessary for great musical achievement is indicated by a recent study of the families of seventy-two famous living opera stars and instrumental artists. It showed that musical talent was possessed by 56 per cent of their mothers, 69 per cent of their fathers and 46 per cent of their brothers and sisters." — From a magazine.

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CHAPTER 4*

ESSENTIALS OF LOGICAL REASONING, II:

PROPOSITIONAL FUNCTIONS, GENERAL PROPOSITIONS, CLASSES

4–1 Introduction. In the preceding chapter we considered some fundamental concepts and rules of deductive logic, mainly from the standpoint of particular propositions. By propositions we meant statements which could be classified (either actually or hypothetically) as "true" or "false" and not both. By *particular* propositions we meant those which avoided use of words like "all," "any," "some." Propositions using words of this nature we referred to as *general* propositions, and we turn our attention to these in this chapter.

As a preliminary example of a general proposition, consider the following statement, which might be made in ordinary conversation by some person prone to generalizing: "all lawyers are good reasoners." Let us try to rewrite this in a form more like our p, q propositions of Chapter 3. We can do this in two ways, which we list along with the original statement for purposes of comparison:

(1) All lawyers are good reasoners.

(2) In all cases, if a person is a lawyer then that person is a good reasoner.

(3) Always "a person is a lawyer" implies "that person is a good reasoner."

At first glance we may think that propositions (2) and (3) are not unlike the familiar implication form, $p \rightarrow q$. However, there are two differences. First, the word "always" or the phrase "in all cases" indicates that more is being asserted than just an implication. Second, the p and q here could not themselves be propositions in our previous sense, for p would have to be "a person is a lawyer" and q "that person is a good reasoner." And how could we determine whether it is true or false that an unspecified person is a lawyer, or whether the same unspecified person is a good reasoner? Clearly

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* This chapter may be omitted without loss of continuity.

this is asking too much. Hence it is misleading to use the standard symbols p, q here, suggesting that the indicated statements are genuine propositions.

In order to understand the true nature of proposition (2) or (3) it will be necessary to introduce the concepts of *propositional function* and *quantified propositions*. This we do in the first part of this chapter. Discussion of these topics will lead into the important topic of *classes*. The connection can be hinted at by going back to proposition (1), which can be interpreted as claiming that the class of all lawyers is contained in the class of good reasoners.

All of the topics mentioned will be found to have considerable usefulness for a more thorough understanding of mathematical theorems and proofs. The chapter will conclude with sections on rules of inference and fallacies involving classes.

4–2 Propositional functions. Consider again the ambiguous statement,

a person is a lawyer,

which is a component of proposition (3) in the preceding section. Let us *replace* "a person" by the symbol x, but without intending x necessarily to *designate* "a person." Then we obtain the following:

x is a lawyer.

This is even more ambiguous than before. We can expect to be able to assign different meanings to the variable term x, for some of which "x is a lawyer" will be true and for others false. Thus if x stands for Justice Black of the United States Supreme Court, it will be true; if x stands for Albert Einstein (the theoretical physicist), it will be false. For other meanings of x only nonsense may result.

By analogy with mathematical terminology we refer to "x is a lawyer" or any similar ambiguous statement as a *propositional function*. Other examples are

x is a right triangle, and x is an isosceles triangle,
x is fond of y,
x is greater than y, or x is equal to y,
x is red implies both that y is white and z is blue,
x is parallel to y, or z is not perpendicular to w.

The concept of propositional function can be defined more precisely as follows:

DEFINITION. A *propositional function* is a statement containing one or more variable terms which becomes a proposition as a result of substituting suitable meanings for the variable or variables.*

It will be convenient to continue the mathematical terminology, and say that if "values" are given to the variables in a propositional function, we obtain "values" of the function, the latter being propositions. Values of the variables which, on substitution in the function, yield *true* propositions are said to *satisfy* the function. (Here the mathematical analogy is not very good, since we ordinarily say that a value of x satisfies an *equation*, not a *function*.)

As suggested by the examples just before the definition, we can build up composite propositional functions from simpler ones by use of the same connectives used for combining propositions, such as "not," "or," "and," "implies," "is equivalent to." Thus suppose we introduce the following notation for the indicated propositional functions:

l_x: x is a lawyer,
r_x: x is a good reasoner.

Then we easily obtain such composite functions as the following:

$l_x \vee r_x$: "x is a lawyer" *or* "x is a good reasoner";
$\sim l_x$: x is not a lawyer;
$l_x \cdot (\sim r_x)$: "x is a lawyer" *and* "x is not a good reasoner."

It is understood of course that, as usual, in making substitutions for x in any function the same value of x is to be used for each occurrence of x throughout the function in question.

We can also make a new propositional function in the form of an implication:

$$l_x \rightarrow r_x,$$

that is,

"x is a lawyer" implies "x is a good reasoner."

This is a propositional *function*, not a proposition, because it is still an ambiguous statement which cannot be judged true or false until a value for x is specified.

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* Strictly speaking, according to this definition any statement like $p \vee q$, $p \cdot q$, $p \rightarrow q$, where p, q are variable or unspecified propositions, is a propositional function *of propositions*. We have previously thought of such statements as composite propositions, however, and it seems preferable to continue to do so. When we refer to a "propositional function" hereafter we shall mean one which is not a function of variable propositions.

If we substitute *Justice Black* for x, then our propositional function becomes the following proposition:

"Justice Black is a lawyer" implies "Justice Black is a good reasoner."

This is a standard implication which could be judged true or false in the usual manner. It would be true (by our definition of *implication*) if we know either that Justice Black is *not* a lawyer (which is contrary to fact), or that he is a good reasoner; otherwise it would be judged false.

If we substitute *Albert Einstein* for x, we obtain

"Albert Einstein is a lawyer" implies "Albert Einstein is a good reasoner."

This is clearly a true proposition, since we know that Einstein is *not* a lawyer (regardless of whether or not he is a good reasoner).

Now imagine that for all possible values of x we find that the propositional function $l_x \rightarrow r_x$ yields a true proposition. This is precisely what is meant by claiming that proposition (1), or (2), or (3), in the preceding section is true — that "all lawyers are good reasoners."

Of course it may turn out that the function $l_x \rightarrow r_x$ is satisfied by some values of x and fails to be satisfied by others.

In the same way, there might be propositional functions which are satisfied by *no* values of x, that is, which always yield false propositions. For example, suppose we use k_x to stand for the indicated composite propositional function,

$$k_x\colon l_x(\sim l_x).$$

Then surely k_x will be false for all possible values of x, since k_x will say

"x is a lawyer" *and* "x is not a lawyer."

These ideas lead up to the subject of quantification, to be discussed in the next section.

The notation used for propositional functions is sometimes the same as for ordinary mathematical functions, e.g., $f(x)$. The subscript notation f_x, however, seems more suggestive of the fact that a *proposition* results by substituting a value of x in the function. If the value $x = a$ is substituted in the function, we naturally use f_a to designate the resulting proposition.

EXERCISES

1. Given the following propositional functions:

h_x: x is in the Western hemisphere;
l_x: x is a living object;
m_x: x is a mineral.

Decide whether the following propositional functions are satisfied when x stands for "the Rock of Gibraltar":

$h_x \vee l_x$, $h_x \cdot m_x$, $h_x \rightarrow l_x$, $l_x \vee m_x$, $l_x \cdot m_x$, $m_x \rightarrow l_x$, $\sim l_x$, $l_x \rightarrow m_x$.

(Note the analogy here with the game of Twenty Questions.)

2. Suppose f_x, g_x are arbitrary propositional functions. What can be said about the *frequency* with which values of x will satisfy the following functions?

(a) $f_x \vee (\sim f_x)$.
(b) $f_x \rightarrow g_x(\sim g_x)$.
(c) $f_x \longleftrightarrow \sim(\sim f_x)$.
(d) $g_x \longleftrightarrow (g_x \vee g_x)$.
(e) $(f_x \cdot g_x)[\sim(f_x \vee g_x)]$.
(f) $(f_x \rightarrow g_x) \longleftrightarrow [(\sim g_x) \rightarrow (\sim f_x)]$.

3. If l_x stands for "x is a lawyer" and r_x for "x is a good reasoner," how could you describe the following?

(a) the x values such that l_x is true;
(b) the x values which satisfy $\sim l_x$;
(c) the x values which satisfy $l_x \vee r_x$;
(d) the x values which satisfy $l_x \rightarrow r_x$;
(e) the x values which satisfy $l_x \cdot r_x$..

4. Suppose that the propositional function $f_x \rightarrow g_x$ is true when $x = a$. Is it conceivable that the following propositions are true?

$\sim g_a$, $f_a \cdot (\sim g_a)$, $(\sim f_a)g_a$, $\sim(f_b \rightarrow g_b)$, $f_a(\sim g_b)$, $\sim(f_a \vee g_a)$.

4–3 Quantified propositions. Given a propositional function f_x we have seen that there are three possibilities: (a) the function f_x may be satisfied by *all* significant values of x, (b) by *some* value or values (but perhaps not all values) of x, or (c) by *no* values of x. It is convenient to have a special notation for the first two of these possibilities, as follows:

(a) $(x)f_x$, which may be read "*for all values* of x, f_x is true," or "*all values* of x satisfy f_x."

(b) $(\exists x)f_x$, which may be read "*for some value* of x, f_x is true," or "*some values* of x satisfy f_x," or "*there exists* a value of x such that f_x is true." (Note that no more than *one value* is required.)

The symbols (x) and $(\exists x)$ are known as *quantifiers;* the former as the *universal quantifier* and the latter as the *existential quantifier.*

Similarly, we might introduce universal and existential quantifiers for propositional functions of more than one variable.

The propositions $(x)f_x$, and $(\exists x)f_x$ are called *quantified propositions;* the former is a *universal proposition* and the latter an *existential proposition.* If a quantified proposition is claimed to be true, no reference is made to specific cases (i.e., specific values of x) as such, even though consideration of specific cases might be involved in justifying the claim.

Often the function f_x used in $(x)f_x$ and $(\exists x)f_x$ will be a composite propositional function, in which case it may be advisable to enclose it in parentheses to avoid confusion. Thus our illustrative general proposition (1) "all lawyers are good reasoners," or (2) "in all cases, if a person is a lawyer, then that person is a good reasoner," can now be written in the form

$$(x)(l_x \rightarrow r_x),$$

using l_x and r_x as in the preceding section.

The third possibility mentioned above, that *no* value of x satisfies f_x, requires no new notation. For this possibility simply indicates that it is *not true* that there exist values of x which satisfy f_x. Hence we can use the following notation:

(c) $\sim[(\exists x)f_x]$, which may be read "there do not exist any values of x such that f_x is true."

In the same way a negation sign could be used in front of a universally quantified proposition obtaining

$\sim[(x)f_x]$, which would mean "it is not true that f_x is satisfied for all x values."

Further consideration of the use of the negation sign in connection with quantifiers will be given in the next section.

Up to this point, the discussion in this chapter may seem to have been somewhat remote from mathematics. To show that the concepts of propositional function and quantified propositions really are significant in mathematical reasoning, consider the following theorem of Euclidean geometry:

If a triangle is equilateral, then it is equiangular.

At first glance, this might seem to say merely:

If x is an equilateral triangle, then x is an equiangular triangle.

The latter is of the same structure as the statement:

If x is a lawyer, then x is a good reasoner.

This statement, lacking the use of a quantifier for generality, is a propositional function, not a proposition. On the other hand, the equilateral triangle theorem is in a different category because it is capable of justification for *any* x. The proof of the theorem, of course, does not bother with the possibility that an x is *not* an equilateral triangle, for in that case the implication holds by definition. The proof is carried through by assuming an *arbitrary* equilateral triangle and showing that the indicated conclusion is true for it (assuming also that the postulates of Euclidean geometry are true). In other words, what is really proved is more accurately expressed as follows:

> For *any* x, if x is an equilateral triangle, then x is an equiangular triangle.

Having proved that the theorem holds for an arbitrary equilateral triangle, it is then tacitly understood that it holds for *all* equilateral triangles, and the theorem is so interpreted thereafter. To express this meaning symbolically, of course, requires use of the universal quantifier. Let us use the following notation:

p_x: x is an equilateral triangle,

q_x: x is an equiangular triangle.

Then the theorem becomes:

$$(x)(p_x \rightarrow q_x),$$

which amounts to the following:

> In all cases, if a triangle is equilateral then it is equiangular.

The latter in turn can be restated in the form:

> All equilateral triangles are equiangular triangles.

(We could have thought of p_x as the conjunction of the two propositional functions "x is a triangle," "x is equilateral," and similarly for q_x. The above form seems more useful here.) We note that the theorem can also be considered to state a necessary (or sufficient) condition. From this point of view the quantified proposition

$$(x)(p_x \rightarrow q_x)$$

can be interpreted as follows, where "in all cases" is implicitly understood:

> A necessary condition that a triangle be equilateral is that it be equiangular.

We chose a more specific wording for stating necessary or sufficient conditions in the preceding chapter (for example, using "triangle ABC" instead of "a triangle") to permit thinking of hypothesis (p) and conclusion (q) as ordinary propositions rather than ambiguous statements like "a triangle is equilateral." The quantified version of the theorem takes care of this difficulty.

The great majority of theorems and many postulates and definitions in mathematics involve quantification in some form, at least implicitly. Often it is found that both types of quantification are used at the same time, and in connection with several variables. This is especially the case in certain advanced branches of mathematics, such as theory of functions, where quite complicated quantification patterns are common. For a simple illustration, consider the following geometric "existence theorem" (sometimes taken as a postulate in high school textbooks):

Every angle has a bisector.

In partial symbolic form this can be written

$$(x)\{x \text{ is an angle} \rightarrow [(\exists y)(y \text{ is the bisector of } x)]\},$$

or it can be consolidated as follows:

$$(x)\{n_x \rightarrow [(\exists y)b_{yx}]\}.$$

Here n_x is "x is an angle" and b_{yx} is "y is the bisector of x" — a propositional function of two variables.

Other illustrations of quantification in mathematical theorems will be given in Exercise 2 below.

Exercises

1. Let r_x, s_x stand for the propositional functions indicated:

r_x: x is a college student,
s_x: x is a person interested in football.

Interpret in words the meaning of the following quantified propositions. (Reword in more idiomatic form if it seems desirable.)

(a) $(x)(r_x \rightarrow s_x)$.
(b) $(\exists x)(\sim r_x) \cdot s_x$.
(c) $(x)(\sim r_x \rightarrow \sim s_x)$.
(d) $(\exists x)[r_x \vee (\sim s_x)]$.

2. Express the following mathematical theorems as quantified propositions (using the most general interpretation of the theorems that is customary):

(a) If a triangle is equilateral, then it is equiangular.

(b) The sum of the angles of any triangle equals 180°.

(c) There exists no triangle containing two right angles (Euclidean geometry).

(d) There exist triangles containing three right angles (Riemannian geometry).

(e) Every right angle can be trisected by ruler and compasses.

(f) Not every angle can be trisected by ruler and compasses.

(g) A necessary condition that an integer be divisible by 10 is that it be divisible by 5.

(h) A necessary and sufficient condition that an integer be divisible by 10 is that its units digit be 0.

(i) A sufficient condition that a quadrilateral be a rectangle is that it be a square.

(j) There exists a rational root for the equation $2x + 3 = 8$.

4–4 Rules of inference for quantified propositions. We shall assume that the rules of inference, and related rules, of the preceding chapter remain valid when the propositions (such as p, q, r) involved in the rules are changed from ordinary propositions to quantified propositions. Such an assumption is justified by the logical development given in *Principia Mathematica*, if we are willing to accept the postulates of logic adopted by Whitehead and Russell.

There is one school of thought in logic and foundations of mathematics — known as the "intuitionist school" — which would not be willing to go along with this. For example, members of this school of thought question the complete validity of the *law of excluded middle*, that a proposition must be either true or false, for quantified propositions. We shall come back to this briefly later (see Section 11–8).

It is still necessary to have some special rules of inference for quantified propositions. Here we shall mention only a few of these, beginning with two rules, those for generalization and specialization.

RULE Q-1. Generalization. *If the propositional function f_x is satisfied by any arbitrary value of x (chosen from the significant values for the function), then f_x is true for all significant values of x, that is, $(x)f_x$ is true.*

RULE Q-2. Specialization. *If $(x)f_x$ is true, and "a" is a significant value for x in the function f_x, then f_a is true.*

In mathematical proofs the f_x will usually be an implication, such as $p_x \rightarrow q_x$.

We note that Rule Q-1 (a postulate in *Principia Mathematica*) justifies the tacit understanding mentioned in the preceding section, that a proof for an arbitrary equilateral triangle establishes the theorem *in all cases*. Incidentally, it should help dispel the outdated idea that deductive reasoning concerns itself only with reasoning from the general to the particular, and that inductive methods are needed to go from the particular to the general.

Rule Q-2 is used continually (but most of the time unconsciously!) when, in the course of proving a mathematical theorem, a previous theorem is invoked. For example, suppose we need to prove a certain triangle ABC equiangular as an essential part of the proof of another theorem. Let us use p_x and q_x as in the preceding section to mean the following:

p_x: x is an equilateral triangle,
q_x: x is an equiangular triangle.

Then the detailed pattern of inference might be like this:

$$
\begin{array}{lll}
(x) & (p_x \rightarrow q_x) & \text{by a previous theorem,} \\
\therefore & p_{ABC} \rightarrow q_{ABC} & \text{by the specialization rule,} \\
 & p_{ABC} & \text{by a previous step in the proof being developed,} \\
\hline
\therefore & q_{ABC} & \text{by the fundamental rule of inference.}
\end{array}
$$

The essential part of the pattern remains the fundamental rule of inference.

Or suppose we wish to claim in another case that a certain triangle DEF is *not* equilateral because the necessary condition for an equilateral triangle is not satisfied, that is, because triangle DEF is not equiangular. Here we would have the following pattern:

$$
\begin{array}{lll}
(x) & p_x \rightarrow q_x & \\
\therefore & p_{DEF} \rightarrow q_{DEF} & \text{by the specialization rule,} \\
 & \sim q_{DEF} & \\
\hline
\therefore & \sim p_{DEF} & \text{by the opposite converse rule.}
\end{array}
$$

An alternate procedure would be to deduce the quantified opposite converse theorem from the given theorem first and then specialize later.

It is instructive to note the phrasing above that "the necessary condition for an equilateral triangle is not satisfied." This shows at once the connection with propositional functions. For the quantified version of the theorem tells us that *for all x's* a necessary condi-

tion that p_x be true is that q_x be true; p_x and q_x are propositional functions which may or may not be satisfied by certain values of x. The failure of q_x to be satisfied when x stands for the particular triangle DEF yields the conclusion that p_x cannot be satisfied when x stands for DEF.

In a similar manner, it is easy to see that if a certain propositional function f_x is a sufficient condition for another propositional function g_x for all values of x, then from the information that a particular value, $x = a$, does satisfy the sufficient condition (f_x) we obtain the result that a satisfies g_x also; that is, g_a is true.

Next we mention two special equivalence rules connecting negation and quantification. Consider first the meaning to be attached to

$$\sim[(x)f_x].$$

This says "it is false that f_x is satisfied by all values of x," which is equivalent to saying that there must be some value or values of x for which f_x fails to be satisfied, that is,

$$(\exists x)(\sim f_x).$$

Similarly,

$$\sim[(\exists x)f_x]$$

asserts "it is false that for some value of x, f_x is true," which is equivalent to saying that for all x values f_x is false, that is,

$$(x)(\sim f_x).$$

The two rules can be stated symbolically as follows:

RULE Q-3. $\sim[(x)f_x] \longleftrightarrow (\exists x)(\sim f_x)$.

RULE Q-4. $\sim[(\exists x)f_x] \longleftrightarrow (x)(\sim f_x)$.

In effect, Rule Q-3 simply asserts that *one* exceptional case disproves a general rule (contrary to the usual interpretation of the familiar saying). Thus Rule Q-3 justifies the claim that an *arbitrary* angle cannot be trisected, for the reason that a *special* angle, e.g., 60°, cannot be trisected.

As an illustration of Rule Q-4 consider the theorem of non-Euclidean geometry (either Lobachevskian or Riemannian) saying that for *any* triangle (and hence *all* triangles) the sum of the angles is *not* 180°. Using the following notation,

f_x: x is a triangle,
g_x: the sum of the angles of x is 180°,

the theorem can be written in either of two equivalent universally quantified forms:

$$(x)(f_x \rightarrow \sim g_x),$$
$$(x)[\sim(f_x \cdot g_x)].$$

The second of these, according to Rule Q-4, is equivalent to

$$\sim[(\exists x)(f_x \cdot g_x)].$$

This asserts the following:

> It is false that there exists an x such that both x is a triangle and the sum of the angles of x is 180°.

In other words, there exists *no* triangle the sum of whose angles is 180°.

Exercises

1. Suppose someone claims that the following proposition is correct:

$(x)(x$ is an angle $\rightarrow x$ cannot be trisected by ruler and compasses),

or

$(x)(p_x \rightarrow \sim q_x)$, where we use the notation:

p_x: x is an angle,
q_x: x can be trisected by ruler and compasses.

Write an existential proposition stating the falsity of the above, and show how to verify it.

2. Suppose s stands for the Euclidean theorem asserting that the sum of the angles of every triangle is equal to 180°, and t stands for the non-Euclidean theorem asserting that for every triangle the sum of the angles is *not* equal to 180°. How does t differ from $\sim s$? How does s differ from $\sim t$?

3. If p stands for $(\exists x)f_x$, verify by using Rules Q-3 and Q-4 that $\sim(\sim p)$ is logically equivalent to p.

4. Suppose a general theorem has been proved (in Euclidean geometry) asserting that a sufficient condition that a parallelogram be a rectangle is that its diagonals be equal. State this symbolically in quantified form and show in detail what rules of inference are involved in applying the theorem to a particular proof that parallelogram $ABCD$ is a rectangle. The following notation is suggested:

p_x: x is a parallelogram,
d_x: the diagonals of x are equal,
r_x: x is a rectangle.

(It may be desirable first to reword the sufficient condition in the form of a conjunction.)

5. Put the following proposition in symbolic form, and prove or disprove it: "All prime integers are odd integers."

6. The United States Constitution contains the following provision: "No person shall be eligible for the office of president who has not attained the age of 35 years." State this both as a positively quantified and negatively quantified proposition. Does it give a necessary condition or sufficient condition for eligibility for the office of president?

7. Distinguish between $(x)(f_x \to \sim g_x)$ and $(x)[\sim(f_x \to g_x)]$.

4–5 Classes. We are now ready to show how the concept of a *class* can be defined by use of the concept of propositional function.

For this purpose let us consider once again the propositional function

$$l_x: x \text{ is a lawyer.}$$

Any x which satisfies l_x is surely a lawyer, and any x which does not satisfy l_x is not a lawyer. It would be reasonable to define the *class* of lawyers as the totality of x values which satisfy l_x.

In general we define a class as follows:

DEFINITION 1. The totality of x *values* which satisfy a given propositional function f_x is called the *class* (*set or collection*) determined by f_x.

The x values which satisfy f_x are called the *members* of the class, or the *elements*, *objects*, *terms*, or *individuals* belonging to the class.

The class determined by f_x will be symbolized by $\hat{x}(f_x)$.

For convenience, we may use the symbol A to stand for the class $\hat{x}(f_x)$. If a certain x value, say $x = a$, satisfies f_x, we say a is a member of the class A, or a belongs to A and we write symbolically:

$$a \in A.$$

Let A and B be the classes determined by f_x and g_x, respectively. Suppose every x value which satisfies f_x satisfies g_x, and conversely; then A and B have precisely the same membership, and we write $A = B$ (meaning here A is identical with B). Otherwise the classes A and B are not identical and we write $A \neq B$. For reference we make the following formal definition.

DEFINITION 2. Two classes A and B are said to be *equal* if they have identically the same membership.

Composite classes can be formed in various ways corresponding to the usual types of composite propositional functions. For example, let us introduce again the function

$$r_x\text{: } x \text{ is a good reasoner,}$$

and consider the following list of classes:

$\hat{x}(l_x \vee r_x)$: the class of individuals who are lawyers or good reasoners;

$\hat{x}(l_x \cdot r_x)$: the class of individuals who are both lawyers and good reasoners;

$\hat{x}[l_x \cdot (\sim r_x)]$: the class of individuals who are lawyers but are not good reasoners;

$\hat{x}[(\sim l_x) \cdot r_x]$: the class of individuals who are not lawyers but are good reasoners;

$\hat{x}[(\sim l_x)(\sim r_x)]$: the class of individuals who are neither lawyers nor good reasoners.

These illustrations give a background for defining *logical sum*, *logical product* of two classes, and *complement* of a given class.

DEFINITION 3. If A and B are two classes determined respectively by the propositional functions f_x and g_x, the *logical sum* of A and B (written $A \vee B$ hereafter *) is the class of individuals satisfying $f_x \vee g_x$. The *logical product* of A and B (written $A \cdot B$, or AB†) is the class of individuals satisfying $f_x \cdot g_x$.

Symbolically, we write this definition:

$$A \vee B = \hat{x}(f_x \vee g_x),$$
$$A \cdot B = \hat{x}(f_x \cdot g_x).$$

The logical sum of two classes is also known as the *union* or *join* of the classes. This is appropriate because the logical sum of A and B must consist of all individuals that satisfy the disjunction $f_x \vee g_x$, and hence of all individuals that are members of A or members of B (including, of course, members of both). In other words, $A \vee B$ is formed by throwing the membership of the two classes together.

The logical product of two classes is also known as the *intersection* or *meet* of the classes. Again, this is appropriate because the logical

* Some writers use $A \cup B$ or $A + B$ for the logical sum of classes A and B. We reserve the plus sign for another operation on classes, to be introduced in Chapter 9. $A \vee B$ may be read "A wedge B."

† Sometimes $A \vee B$ or $A \cap B$.

product of A and B must consist of all individuals that satisfy the conjunction $f_x \cdot g_x$, and hence of all individuals that are members both of A and B simultaneously.

Next we introduce the *complement* of a class.

DEFINITION 4. If A is a class determined by the propositional function f_x, then the *complement* (or *negation*) of A (written A') is the class determined by $\sim f_x$.

In this definition, as usual, the x values to be considered are those and only those for which f_x can be judged to be significant.

Let us investigate next the class

$$A \vee A',$$

which is the same as the class

$$\hat{x}[f_x \vee (\sim f_x)].$$

By the agreement just mentioned the x values to be considered are precisely those for which

$$f_x \text{ is true} \quad \text{or} \quad f_x \text{ is false},$$

that is, for which the function $[f_x \vee (\sim f_x)]$ is satisfied. This means that the class $A \vee A'$ consists of the *totality* of x values for which f_x is significant.

For this reason, we say that $A \vee A'$ is the *universal class*, relative to the defining propositional function f_x. The universal class will be designated by the symbol, U.

The class $A \cdot A'$ is of comparable significance. The defining function here is

$$f_x(\sim f_x).$$

But for every x for which f_x becomes a significant proposition we know that $f_x(\sim f_x)$ is false. Thus there are *no* x values satisfying the function, and we call $A \cdot A'$ the *empty class*, the *null class*, or the *zero class*, relative to the defining function f_x. This will be designated by the symbol Z.

To summarize, we write for all classes A the equalities

$$A \vee A' = U,$$
$$A \cdot A' = Z.$$

These equalities will be found to be fundamental in the abstract mathematical theory of classes known as algebra of classes, or still more abstractly as a Boolean algebra. (In Chapter 9 we develop the

theory of Boolean algebras as a useful example of a postulational system.)

There are quite possibly other classes besides A', the complement of A, which have no elements in common with A. Suppose B is such a class. Then the logical product of A and B is the null class. This leads to the following definition:

DEFINITION 5. Two classes which have no members in common are called *mutually exclusive*, or *disjoint*, classes.

Accordingly, A and B are disjoint classes if and only if

$$AB = Z.$$

For a familiar example of disjoint classes, let A be the class of all even integers, and let B be the class of all odd integers.

If two classes A and B are *not* disjoint, we can write, of course,

$$AB \neq Z.$$

This means that A and B have *some* members (at least one) in common. In other words, there exists an individual c such that

$$c \in A \quad \text{and} \quad c \in B.$$

If $A = x(f_x)$ and $B = x(g_x)$, the same information is conveyed by the quantified proposition:

$$(\exists x)(f_x \cdot g_x).$$

Propositions asserting that two classes have some members in common are of frequent occurrence. They often appear in the following form:

Some lawyers are good reasoners.
Some college students are football players.
Some isosceles triangles are equilateral triangles.

Although the order in which the classes occur in each of these propositions may appear significant, actually nothing more is claimed than that some members of one class belong to another class. Thus the propositions are reversible, yielding:

Some good reasoners are lawyers.
Some football players are college students.
Some equilateral triangles are isosceles triangles.

From the geometric standpoint, we can strengthen the last example as follows:

All equilateral triangles are isosceles triangles.

This statement says that every member of the class of equilateral triangles is a member of the class of isosceles triangles, or more concisely, the class of equilateral triangles *is contained in* the class of isosceles triangles.

Similarly, our example at the beginning of the chapter makes the claim that the class of lawyers is contained in the class of good reasoners.

It is desirable to insert here a formal definition of the phrase "contained in."

DEFINITION 6. Given the classes A and B; if every member of class A is also a member of B, then we say that class A *is contained in* class B, or *is included in* class B. Also we say that A is a *subclass* of B, or that B *contains* A.

Symbolically, we write $A \subset B$ (or $B \supset A$) *to designate that class* A *is contained in class* B.

We can now write the following equivalence:

$$(A \subset B) \longleftrightarrow [(x)\{(x \in A) \rightarrow (x \in B)\}].$$

If we assume that A and B have the defining functions f_x and g_x, respectively, this can be changed to

$$(A \subset B) \longleftrightarrow [(x)(f_x \rightarrow g_x)].$$

The latter says that "class A is contained in class B" is equivalent to "everything which has the defining property for class A also has the defining property for class B." Briefly, we can read $A \subset B$ as "all A's are B's," or, equally well, "only B's are A's." Of course, this also says that a *necessary* condition for membership in class A is membership in class B.

According to the definition, we see that any class is contained in itself, so that we can always write

$$A \subset A.$$

Furthermore,

$$\text{if } A \subset B \quad \text{and} \quad B \subset A, \quad \text{then} \quad A = B,$$

since the membership of A and B must be the same. An example of this is supplied by taking A and B respectively as the classes of equilateral and equiangular triangles.

On the other hand, if $A \subset B$ and it is false that $B \subset A$, then there must exist at least one member of B which does not belong to A.

In this case A is called a *proper* subclass of B. For an example here, take A again as the class of equilateral triangles but B as the class of isosceles triangles.

In case class A = class B, the symbols A and B can be used interchangeably, just as we can substitute *equivalent* propositions for each other in the manner agreed to in Chapter 3.

In fact, the *proposition* $A = B$ is itself equivalent to the conjunction

$$A \subset B \quad \textit{and} \quad B \subset A,$$

and thus to the conjunction

$$(x)(f_x \rightarrow g_x) \quad \textit{and} \quad (x)(g_x \rightarrow f_x),$$

and hence is equivalent to the proposition that

$$(x)(f_x \longleftrightarrow g_x).$$

Similarly an *equivalence* of propositions often leads to a corresponding *equality* (in the sense of *identity*) for classes. For example, consider the equivalence

$$(p \vee q) \longleftrightarrow (q \vee p).$$

This permits us to write

$$(x)[(f_x \vee g_x) \longleftrightarrow (g_x \vee f_x)].$$

But $f_x \vee g_x$ is the defining function for $A \vee B$ and $(g_x \vee f_x)$ is the defining function for $B \vee A$. Thus we have the equality

$$A \vee B = B \vee A.$$

Of course, we could easily have convinced ourselves that this is a true equality without going all the way back to the equivalent propositions $p \vee q$ and $q \vee p$. Here, however, we are interested in emphasizing the parallelism between equivalent propositions and equal classes, and the underlying reasons for the parallelism.

We conclude this section with a list of some dual equalities of classes which will be helpful later on in connection with our development of Boolean algebra.

The first four equalities in each column correspond precisely (in the manner illustrated above) to the equivalences listed for propositions in Section 3–10. These include commutative, associative, and distributive properties. Also included are the analogs for classes of the double negation principle and DeMorgan's laws, together with special equalities involving the null class and the universal class.

1. $A \vee A = A$.	1′. $AA = A$.
2. $A \vee B = B \vee A$.	2′. $AB = BA$.
3. $(A \vee B) \vee C = A \vee (B \vee C)$.	3′. $(AB)C = A(BC)$.
4. $A(B \vee C) = AB \vee AC$.	4′. $A \vee BC = (A \vee B)(A \vee C)$.
5. $(A')' = A$.	
6. $(A \vee B)' = A'B'$.	6′. $(AB)' = A' \vee B'$. (DeMorgan's laws for classes)
7. $A \vee Z = A$.	7′. $AU = A$.
8. $A \vee A' = U$.	8′. $AA' = Z$.

EXERCISES

1. Distinguish between the slogans "*all* the news that's fit to print" and "*only* news that's fit to print."

2. Three advanced college courses designated respectively by A, B, C have the following membership rolls:

A	B	C
Barton	Barton	Barton
Clements	Fulton	Meyer
Fulton	Hatch	Quincy
Hatch	Meyer	
Perkins	Quincy	
Stafford		

(a) Give the membership of the following "classes": $A \vee B$, AB, $A'C$, $(AB)C'$, $A(B \vee C)$.

(b) Write and interpret equalities for BC and $B \vee C$.

3. If A is the class of all negative integers, B the class of all positive integers, and C the class of all positive square integers, interpret the following: $A \vee B$, AB, BC, $B \vee C$.

4. Express as necessary, sufficient, or necessary *and* sufficient conditions:

(a) All rectangles are quadrilaterals.

(b) Only isosceles triangles are equilateral triangles.

(c) Only rectangles are squares.

(d) All triangles and only those triangles that are equilateral are equiangular.

(e) All points and only those points whose coordinates satisfy the equation $x^2 + y^2 = r^2$ belong to the locus of the equation.

(f) Only integers divisible by 3 are divisible by 9.

5. Express each of the statements of the preceding exercise as an inclusion or equality of classes.

6. Show that $A \subset B$ is equivalent to $A \vee B = B$ and also to $AB = A$, by using the concept of class membership. [Cf. Exercise 2(b).]

4–6 Rules of inference for classes. There are a number of useful rules of inference involving class inclusion which are often analogous to the rules of Chapter 3 involving implication of propositions. Here we state a few of the most common rules (we use the symbol $\notin$ to mean "does not belong to" and $\neq$ to mean "is not equal"):

RULE C-1.
$$\begin{array}{l} A \subset B \\ \underline{x \in A} \\ \therefore\ x \in B \end{array}$$

RULE C-2.
$$\begin{array}{l} A \subset B \\ \underline{B \subset C} \\ \therefore\ A \subset C \end{array}$$

RULE C-3.
$$\begin{array}{l} A \subset B \\ \underline{x \notin B} \\ \therefore\ x \notin A \end{array}$$

RULE C-4.
$$\begin{array}{l} A \subset B \\ \hline \therefore\ B' \subset A' \end{array}$$

RULE C-5.
$$\begin{array}{l} AB \neq Z \\ \underline{B \subset C} \\ \therefore\ AC \neq Z \end{array}$$

These can easily be verified either by direct consideration of the meaning of class inclusion and class membership or by changing the class inclusions to quantified propositions. If we do the latter in Rule C-1, using f_x and g_x as respective defining functions for A and B and designating the x (which belongs to A) in the second step of the rule by the symbol a, we get:

$$\begin{array}{ll} & (x)(f_x \rightarrow g_x) \\ & \underline{f_a} \qquad\qquad\qquad\qquad (\text{i.e., } a \in A) \\ \therefore & g_a \qquad\qquad\qquad\qquad (\text{i.e., } a \in B). \end{array}$$

This is almost a repetition of the pattern obtained in Section 4–4 for applying a "previous" general theorem in a particular case. The only difference is that the step from

$$(x)(f_x \rightarrow g_x)$$

to

$$f_a \rightarrow g_a$$

is deleted. Thus Rule C-1 is an adaptation of the fundamental rule of inference.

We could also justify Rule C-1 directly by the concepts of class inclusion and class membership. Thus we now assume we are given

$$(x)[(x \in A) \rightarrow (x \in B)],$$

and

$$a \in A.$$

These yield the particular implication

$$(a \in A) \rightarrow (a \in B),$$

but again $a \in A$;

hence $a \in B$ by the fundamental rule.

Rule C-2 is an adaptation of the chain rule. It merely conforms to the following pattern:

$$\begin{array}{l} (x)(f_x \rightarrow g_x) \\ \underline{(x)(g_x \rightarrow h_x)} \\ \therefore\ (x)(f_x \rightarrow h_x) \end{array}$$

Rules C-1 and C-2 can be restated in the form of traditional syllogisms as follows:

(C-1). All A's are B's.
x is an A.
Hence x is a B.

(C-2). All A's are B's.
All B's are C's.
Hence all A's are C's.

Rule C-1 is perennially illustrated by an instance concerning the mortality of Socrates:

All men are mortal.
Socrates is a man.
Hence Socrates is mortal.

Rule C-2 can be restated in the form

All A's are B's.
All C's are A's.
Hence all C's are B's.

This shows that it is essentially a generalization of Rule C-1.

Rules C-3 and C-4 are easily seen to be connected with the opposite converse rules for propositions, with Rule C-4 a generalization of Rule C-3. Rule C-5 has the existence feature mentioned in the preceding section and yields another form of classical syllogism, namely:

Some A's are B's.
All B's are C's.
Hence some A's are C's.

Finally, we mention two standard syllogistic rules involving disjoint classes:

RULE C-6. $AB = Z$, $x \notin A$, $\therefore\ x \notin B$.

RULE C-7. $AB = Z$, $C \subset A$, $\therefore\ BC = Z$.

Rule C-6 is justified at once from the definition of disjoint classes. The pattern is the same as

No A's are B's.
x is an A.

Hence x is not a B.

The pattern of Rule C-7 is a generalization of this, namely,

No A's are B's.
All C's are A's.

$\therefore$ No C's are B's.

For further illustrations of the use of the various rules see the exercises below.

Exercises

1. Correlate the following arguments with rules of inference for classes:

(a) No prime integers greater than 2 are even.
x is a prime integer greater than 2.

$\therefore$ x is not an even integer.

(b) Some primes are integers divisible by 2.
All integers divisible by 2 are even integers.

$\therefore$ Some primes are even integers.

(c) No equilateral triangles are right triangles.
All equiangular triangles are equilateral triangles.

$\therefore$ No equiangular triangles are right triangles.

(d) Some parallelograms are squares.
All squares are rectangles.

Hence some parallelograms are rectangles.

2. Show how Rule C-3 can be obtained from the opposite converse rule of inference for propositions.

3. Give a proof of Rule C-5, based on the concept of class membership. (Here we are given $AB \neq Z$ and $B \subset C$ and we wish to show $AC \neq Z$.) By hypothesis,

$$(\exists x)[(x \in A) \cdot (x \in B)].$$

Let us designate a specific x belonging to both A and B by the symbol x_1. Then we have

$$x_1 \in A \quad \textit{and} \quad x_1 \in B.$$

But also by hypothesis $B \subset C$, which can be written

$$(x)[(x \in B) \rightarrow (x \in C)].$$

Now the proof can easily be completed by stating that this implication holds for the special value x_1, and then using the fundamental rule of inference to show that x_1 belongs to C as well as to A and B.

4. Express each of the following in standard symbolism and give an illustration of each situation:

All A's are B's.
Not all A's are B's (i.e., it is false that all A's are B's).
Some A's are B's.
No A's are B's.
All non-A's are non-B's.

5. Obtain a conclusion in each of the following cases and state the rule of inference used, whenever one of the numbered rules applies.

(a) No Republicans are New Dealers.
All isolationists are Republicans.

Hence . . . .

(b) All brokers are smokers.
Mr. H is a nonsmoker.

Hence . . . .

(c) No brokers are smokers.
Mr. J is a broker.

Hence . . . .

(d) Not all brokers are smokers.

Hence . . . .

(e) Some politicians are corrupt.
All corrupt persons should be brought to trial.

Hence . . . .

(f) All Socialists are non-Communists.

Hence all Communists are . . .

(g) All squares are rectangles.
All rectangles are parallelograms.
All parallelograms are quadrilaterals.

Hence . . . .

(h) Some negative numbers are rational numbers.
No rational numbers are imaginary.

Hence . . . .

6. Sometimes a particular proposition will occur in combination with a quantified one or a proposition about classes, or the latter types may be combined among themselves. The usual rules can be applied. Obtain conclusions in the following cases:

(a) If Jones is correctly informed, all vegetarians are healthy persons.
Not all vegetarians are healthy persons.
———
Hence . . . .

(b) If all volunteers are efficient, no one will be hired for the job.
Some one will be hired for the job.
———
Hence . . . .

(c) If all lines through P intersect line AB, then there is no parallel to AB through P.
There is a parallel to AB through P.
———
Hence . . . .

(d) If there are any rational roots for the equation $x^2 - 2 = 0$, then either ± 1 or ± 2 are roots.
It is not the case that ± 1 or ± 2 are roots.
———
Hence . . . .

7. Show that the proposition $A \subset B$ is equivalent both to $A' \vee B = U$ and to $AB' = Z$.

Suggestion. Let f_x and g_x be the defining functions for A and B. Then $A \subset B$ is equivalent to

$$(x)(f_x \to g_x),$$

that is,

$$(x)(\sim f_x \vee g_x) \quad \text{by definition of implication.}$$

The complete function in parentheses is the defining function for $A' \vee B$ and is satisfied by *all* x's. Hence the given inclusion is equivalent to

$$A' \vee B = U.$$

For the second part use the equivalence

$$(\sim f_x \vee g_x) \longleftrightarrow \sim[f_x \cdot (\sim g_x)].$$

4–7 Fallacies involving classes. To conclude this chapter, we list below (and cross out) some common fallacies in reasoning involving classes. Two of these (F-1 and F-3) are closely related to the propositional fallacies of assuming the converse, and two others (F-2 and F-4) are similarly related to assuming the opposite. F-3 and F-4 are generalizations of F-1 and F-2 respectively. The fallacious nature of the arguments can be verified either directly or by going back to the defining functions and quantified propositions.

F-1. $A \subset B$
$x \in B$
$\therefore x \in A$

F-2. $A \subset B$
$x \notin A$
$\therefore x \notin B$

F-3. $A \subset B$
$C \subset B$
$\therefore C \subset A$

F-4. $A \subset B$
$\sim[C \subset A]$
$\therefore \sim[C \subset B]$

F-5. $AB \neq Z$
$x \in A$
$\therefore x \in B$

F-6. $AB \neq Z$
$x \notin A$
$\therefore x \notin B$

For illustrations of some of these fallacies consider the following exhibits, which unfortunately have all too familiar a ring.

(Fallacy F-3) All pacifists are supporters of peace by negotiation.
All ministers are supporters of peace by negotiation.

Hence all ministers are pacifists.

(Fallacy F-5) Some government employees are persons capable of being bribed.
X is a government employee.

Hence X is capable of being bribed.

(Fallacy F-6) Some persons who favored recognizing China in 1950 are Communist sympathizers.
Y did not favor recognizing China in 1950.

Hence Y is not a Communist sympathizer.

It is easy to see that arguments of the above nature are especially popular with persons who favor the doctrine of "guilt by association," or perhaps with persons trying to defend themselves against this doctrine.

Arguments conforming to F-4 can sometimes be camouflaged to appear like respectable syllogisms. For example, consider the following:

(Fallacy F-4) All rectangles are parallelograms.
Not all quadrilaterals are rectangles.

Hence not all quadrilaterals are parallelograms.

Each of these three propositions is true in Euclidean geometry. Still, the *argument* is a nonvalid argument, as can be emphasized by keeping the form of the propositions the same, but changing the word "quadrilaterals" to "rhombuses," thus obtaining

(Fallacy F-4) All rectangles are parallelograms.
Not all rhombuses are rectangles.

Hence not all rhombuses are parallelograms.

Here, of course, the first two statements are true but the third is false. The situation can be clarified by recalling that rectangles and rhombuses are overlapping subclasses of parallelograms (or by restating the second premise to read "Some rhombuses are not rectangles" with a similar change in the proposed conclusion).

Many books on logic use diagrams involving circles which overlap, or are within, or fail to intersect other circles to illustrate the corresponding possible relationships of classes. We have refrained from doing that here, partly because of our desire to emphasize that rules of inference, and fallacies for classes, grow out of those for propositions, and partly because it is possible for the use of such diagrams to degenerate into the substitution of a kind of doodling process for thinking.

Exercises

1. Give illustrations of fallacies F-1 and F-6.
2. State the class fallacies directly analogous to assuming that the converse or opposite of a true implication is true.
3. Are the following arguments valid? If not, try to classify the fallacy.

(a) "I think the slogan 'believe in yourself and you're bound to believe in *X* cigarettes' is the first attempt by a cigarette manufacturer to inject the question of mental stability into its ads. However, I believe the company limits its potential sales badly by implying that those who haven't achieved a certain serenity are not likely to buy the weeds." — From a commentator's column.

(b) "The assistant chief of the Visa Division of the State Department said he believes the Joint Anti-Fascist Refugee Committee is 'honeycombed with Communists,' since the name included the words 'anti-Fascist'." — From a United Press dispatch.

4–8 Conclusion. One striking result of our discussion of logical reasoning in these two chapters should be the realization that the various rules of reasoning used continually in mathematics are applicable, and ought to be indispensable, in numerous situations which ordinarily appear remote from mathematics. In fact, it is inherent in the nature of deductive logic, as described at the beginning of Chapter 3, that this be the case. The extreme abstractness of the modern symbolic approach to logic should serve to emphasize this extreme applicability in all kinds of intellectual activities. In the next chapter, we turn to the application of logic in those intellectual activities comprising the domain of science.

CHAPTER 5

SCIENTIFIC METHOD AND SCIENTIFIC KNOWLEDGE

5–1 Introduction and historical view. In the first four chapters we have been dealing with the logical structure of mathematical theories and with the reasoning process used in developing them, together with the applications of the same reasoning process in numerous nonmathematical situations. We turn our attention now to a related topic, namely, the process of formulating scientific theories, including consideration both of the scientific method used in the process and of the nature of scientific knowledge, the end result of the process.

By a *science* we shall here understand a branch of systematized knowledge and active research which is specifically concerned with determination of the truth concerning natural phenomena. This definition may be narrower than is customary nowadays, but will be satisfactory for our purpose. According to this definition, it would hardly be correct to refer to a branch of pure mathematics as a science. A branch of applied mathematics would more nearly qualify, but even here the emphasis is on construction of mathematical theories to be put at the disposal of the sciences, rather than on specific acquisition of natural knowledge.

We begin by asking: How is scientific knowledge in any field acquired? A brief historical view is helpful here. In medieval times information about natural events was usually sought by consulting either the works of the encyclopedic Greek philosopher, Aristotle, or the treatises of the Greek astronomer, Ptolemy, or the doctrines of the Church fathers. The transition to modern science began when men like Roger and Francis Bacon, Copernicus, and Galileo started to question the validity of knowledge based on the assertions of Aristotle and other supposed authorities. These men advocated a more reasonable procedure: in order to obtain knowledge of nature, study nature. They emphasized the need of observation, experimentation, and collection and study of data as a necessary preliminary to the systematizing of knowledge about the real world. The famous explorers and inventors from the time of Columbus onward displayed some of the same tendencies.

It must not be supposed that the transition from medieval reverence for authority to the modern scientific respect for observation and experiment was easily and quickly accomplished; on the contrary, it was a long struggle. The forces which had to be overcome were accumulated mental inertia, superstition, and intolerance. Certainly, in many ways, the study of the history of science should be at least as enlightening as the study of political or social history.

5–2 Scientific method and reflective thinking. Confirmation and infirmation. The modern method of scientific inquiry has two main aspects: *inductive* and *deductive*. The *inductive* aspect is concerned with the search for facts or principles by observation and experimental procedure. It was this aspect which was generally neglected in medieval times. The *deductive* aspect involves, as usual, logical reasoning leading from statements or principles taken as premises to new statements or principles. The mistake in medieval inquiry was to rely almost solely on deduction from principles laid down by the supposed authorities, rather than principles supported by inductive evidence. Today, it is generally recognized that inductive and deductive procedures must complement and reenforce each other in the formulation of scientific knowledge.

The detailed steps in the modern version of scientific method are not essentially different from the steps in any process of *reflective thinking*. A well-known outline of these steps was given by John Dewey, famous American philosopher, in his book, *How We Think*.* (No doubt the book should have been called "How We Ought to Think.")

According to Dewey, before there is any reflective thinking on the part of an individual there must be some kind of difficulty or perplexity experienced by that individual. The difficulty may be a comparatively trivial matter concerned with plans for the day's activities, or it may be a weighty problem concerned with stars or atoms or differential equations. In any case, an end to the state of perplexity is desired. The process of reflective thinking is a systematic process which will supposedly lead to, or at least toward, a satisfactory solution.

Dewey's steps in reflective thinking (originally five in number) are essentially as follows:

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* John Dewey, *How We Think*, D. C. Heath and Company, 1933.

(1) Transformation of the difficulty into a clearly defined *intellectual problem*, taking due account of observed facts and the apparent nature of the difficulty.

(2) Formulation of suggestions, based on observations, or past experience, as *definite hypotheses* which might lead to a solution.

(3) Exploration of the *logical consequences* of the various hypotheses.

(4) *Testing the hypotheses*, either through mental reflection or action, by comparing their logical consequences with the observed facts and thus either deciding on a probable solution or the lack of a solution. If it appears that no solution has been obtained, the need of further application of some of the preceding steps is indicated.

Thus the reflective thinking process might be characterized as a refined process of trial and error. Note that the word "hypothesis" as used here has a somewhat special meaning, different from the "hypothesis of a theorem." Here the idea is that of a tentative assumption.

Step (4), testing the hypotheses, is usually crucial. If the test is favorable to a certain hypothesis, we say that the hypothesis is *confirmed*. If unfavorable, the hypothesis is *infirmed*. In scientific method we find often that *infirmation* yields a more definite answer, though negative, than *confirmation* which may appear to yield a positive answer. The difference in logical pattern between the two cases explains this.

Thus, suppose it is desired to test a certain scientific hypothesis designated as P (which may consist of several principles taken jointly). Let Q stand for certain related observational or experimental facts. Then in case hypothesis P is *confirmed* by Q, we have (in rather oversimplified form)

$$P \rightarrow Q,$$
$$Q,$$

and would like to conclude P. We know that we cannot do this on purely logical grounds without being guilty of the fallacy of assuming the converse. But of course other considerations of an observational or logical nature may point more conclusively to P. Accumulation of a large amount of confirming evidence may suggest that P is the *best* available hypothesis.

On the other hand, if hypothesis P is *infirmed* by the observed facts, this indicates a discrepancy between theory and experiment.

Using R now to designate the logical consequences of P, we have essentially the following pattern:

$$\begin{array}{l} P \rightarrow R \\ \sim R \\ \hline \sim P \end{array}$$

whence

by the opposite converse rule of inference. This is a familiar unit of indirect reasoning.

We must not be misled by our emphasis here on scientific method into imagining that the reflective thinking process is not equally useful for mathematical investigations. Logical systems developed by deductive reasoning occur as more or less perfected end products in mathematics. We have seen that much mathematical research is stimulated by the attempt to solve problems, and a great variety of methods, inductive and intuitive as well as deductive, are considered allowable in the attempt. And of course much mathematical learning in traditional school or college courses in mathematics arises from problem-solving activity. Even the process of formulating a logical proof in geometry can be considered a type of problem.

In this connection we mention a book entitled *How to Solve It* * by G. Polya, a well-known research mathematician. The book might be called a mathematical version of Dewey's steps and should be especially useful to prospective teachers of mathematics.

Exercises

1. Which of Dewey's steps would you call predominantly inductive, which mainly deductive?
2. What is the role of indirect reasoning in the reflective thinking process?
3. From your own daily affairs, find an illustration of a situation where you have used, or might profitably have used, steps similar to Dewey's.
4. Correlate a physician's method of diagnosing a disease with the steps in reflective thinking. (Substitute a mechanic's method in locating car trouble if you prefer.)

5–3 The scientific method in action. Dewey acknowledged that his steps should not be considered hard-and-fast rules of procedure. Rather they should be thought of as suggestive aids to systematizing the whole thinking process. Sometimes it may be that certain steps are emphasized almost to the exclusion of others, and the order in

* G. Polya, *How to Solve It*, Princeton University Press, 1946.

which they occur may be different. It may also be that there is conscious or unconscious division of labor between different thinkers in carrying through the various steps. This often happens in scientific work, and sometimes over a long period of time.

We shall draw on the history of science for a famous astronomical illustration of the scientific method in action. We refer to the discovery of the planet Neptune in the year 1846.

Previous to 1846 the known planets of the solar system were Mercury, Venus, the Earth, Mars, Jupiter, Saturn, and Uranus. The latter had been discovered only as recently as 1781. Its theoretical orbit had been computed by astronomers, using mathematical formulas and rules which relied heavily on Newton's laws of gravitation and motion. These laws had been found adequate to explain and predict the motions of the known planets before the discovery of Uranus. Considered as hypotheses, these principles had stood the test of observational verification within the limits set by the inevitable errors in the use of measuring instruments.

However, after Uranus had been observed for a number of years, discrepancies began to be noticed between its theoretical orbit and its observed path. At first it was probably thought that these discrepancies might be accounted for as errors in measurement, but as time passed, the disagreement between theory and observation became too systematic and sizeable to be explained away by the hypothesis of observational errors. Thus an astronomical difficulty arose. By about 1840 it was considered to be an acute problem and was the subject of serious investigation.

Various hypotheses were considered for solving the mystery. No one at that time was willing to question the validity of Newton's laws, which had stood so much previous observational testing. Assuming that these principles were still valid, astronomers decided that a possible cause might be the existence of another planet as yet unobserved, which by its attraction was causing Uranus to deviate from its computed course. Finally, in 1845, an English astronomer, Adams, and a French astronomer, Leverrier, independently computed the position of a hypothetical planet which would modify the theoretical motion of Uranus sufficiently to bring about close agreement with the observed motion.

Although the theoretical computations of Adams and Leverrier did not check with each other exactly, they were sufficiently close to suggest a definitely limited search of the heavens for the hypothetical

planet. The planet (no brighter than an eighth magnitude or very dim star) was actually spotted by telescope at the Berlin observatory in September 1846. It was verified that the object sighted was something new because it did not appear on recently formulated star charts of this region of the sky. Furthermore, it was verified observationally that the object was a planet and not a "fixed star," due to the fact that its position was found to have changed from one evening to the next.

In this case the confirmation of the hypothesis (of the existence of the planet) was of a more definite nature than is usually possible. Of course the incident gave added confirmation to Newton's laws, but nevertheless Newton's laws were soon destined to lose their uncontested position in astronomy.

Exercises

1. Correlate the steps in the discovery of Neptune with Dewey's steps in reflective thinking.
2. Discuss the roles of *inductive and deductive* methods in the discovery of Neptune. Which seems to be of greater importance?
3. Investigate the process by which the planet Pluto was discovered (in the year 1930).

5–4 On scientific laws. Newton's viewpoint. The remainder of this chapter will be concerned with certain aspects of scientific knowledge including such topics as the following: the nature of scientific laws and theories, with illustrations based on the physical theories of Newton and Einstein; the role of geometry in relation to physical science; and a comparison of scientific and mathematical truth.

In the astronomical illustration of the preceding section we had occasion to notice the importance of Newton's laws. Probably the most famous of these is his law of gravitation. Because of its sweeping scope the adjective "universal" is often applied to it. It has come to be stated about as follows:

> Every object in the universe attracts every other object with a force which varies directly as the product of their masses and inversely as the square of the distance between them.

There are numerous other familiar scientific principles which are commonly referred to as laws. Such principles have certainly been

formulated as a result of scientific investigation and by use of the scientific method, but by what right are they given the rather dogmatic sounding title of "law"?

Let us consider Newton's own point of view toward his law of gravitation and laws of motion. He presented these laws in the treatise entitled *Philosophiae Naturalis Principia Mathematica* (1687) commonly called just the *Principia*.* In this famous work he expounded his system of mechanics, and also his "System of the World." The *Principia* was to Newton as the *Elements* was to Euclid, and Newton actually used the Euclidean pattern of logical organization, based on "books" with axioms, definitions, problems, theorems, corollaries (and many "scholia" or supplementary comments). The laws of motion † occupied the place of honor as axioms (although certainly these were far different from Euclid's "common notions"). The law of gravitation was brought in by stages throughout the work, but did not appear in the concise form quoted above and — contrary to the usual impression — was not an axiom or postulate. In other words, the "law of gravitation" is a summary of various principles proved by Newton in theorems and corollaries.

It would be incorrect to conclude that Newton's *Principia* aimed to be completely theoretical and deductive, like Euclid's *Elements*. On the contrary, Newton was definitely attempting to build a logical system which would reconcile his abstract theory with known experimental data, or with principles and laws (such as Kepler's laws of planetary motion) accepted in the current sciences of physics and astronomy. Moreover, he often incorporated into the *Principia* the results of specific experiments (with due consideration of errors in

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* For a modern edition of Newton's *Principia*, see *Sir Isaac Newton's Mathematical Principles of Natural Philosophy* (Cajori's revision of Motte's translation), University of California Press, 1934.

† Newton's laws of motion as given in the *Principia* (Cajori edition cited in the preceding footnote) are as follows:

Law I. Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.

Law II. The change of motion is proportional to the motive force impressed, and is made in the direction of the right line in which that force is impressed.

Law III. To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

measurement) performed by himself or others for the purpose of checking his theoretical conclusions.

The following quotation seems to give a significant summary of Newton's viewpoint toward scientific method and the formulation of his laws.*

"Hitherto we have explained the phenomena of the heavens and of our sea by the power of gravity, but have not yet assigned the cause of this power . . . I have not been able to discover the cause of those properties of gravity from phenomena, and I frame no hypotheses; for whatever is not deduced from the phenomena is to be called an hypothesis; and hypotheses, whether metaphysical or physical, whether of occult qualities or mechanical . . . have no place in experimental philosophy. In this philosophy particular propositions are inferred from the phenomena, and afterwards rendered general by induction. Thus it was that . . . the laws of motion and gravitation were discovered. And to us it is enough that gravity does really exist and act according to the laws which we have explained, and abundantly serves to account for all the motions of the celestial bodies, and of our sea."

To the extent that this passage is representative of Newton's views, it indicates that he tended toward the absolute truth concept of scientific knowledge. He apparently believed that natural phenomena themselves, together with inductive and deductive methods, can supply true principles or laws, which are to be considered facts, not based on hypotheses. Thus it is significant that he was willing to use the label "axioms" for his laws of motion, even though he took pains to cite experimental evidence in their support. Furthermore, he seemed unaware of the fact that some of his definitions and explanations preceding the axioms (e.g., his concepts of *absolute* time and *absolute* motion) themselves involved elements of assumption (later to be disputed by Einstein).

However, Newton's statement "I frame no hypotheses" is open to misinterpretation. He was simply disclaiming here any desire to speculate as to possible *causes* of the force of gravitation. Surely Newton realized that in his other research activities he often put forth hypotheses for consideration. But he did not believe that the *existence* of gravitation could be a matter of speculation any longer. He was so sure of the laws of gravitation and motion on the basis of

* Newton, *op. cit.*, pp. 546–547.

his version of scientific method that he did not hesitate to remove them from the category of hypotheses.

Newton's idea of trying to avoid all speculation by generalizing propositions *inferred* directly from the phenomena seems somewhat hazy and untenable. For surely even the narrow process of interpreting direct observations can hardly avoid the use of some assumptions — concerned, for example, with such statistical questions as the degree of exhaustiveness or validity of the evidence, and the probable limits of experimental error. What should be said then concerning the more sweeping generalization process Newton advocated? Such a process he apparently justified by the "abundant" confirmation referred to in the last sentence of the quotation.

For many years, physicists and astronomers agreed with Newton that the law of gravitation and related laws were unquestionable facts. We shall see their mistake more vividly in the next section when we discuss the Einstein theory.

It is usually true that principles are not called "laws" unless they are supported by a generous amount of experimental confirmation, or perhaps unless they are deducible from other principles, themselves well established. Some laws may be little more than a statistical generalization of observed relationships. Others, like Newton's laws, may be much more abstract and of such a fundamental character that they are in a position to serve as postulates in a logical organization of a science. In such a case, the acceptability of the postulates may depend on internal structural considerations of the theory based on them, as well as on experimental checking of their logical consequences. But whether they are considered generalizations, axioms, postulates, or theorems, we know that they cannot be considered absolute.

For these reasons it might seem desirable to delete the word "law" from scientific vocabulary. This is not likely to happen. Perhaps the next best thing would be to emphasize that laws in all departments of knowledge, as well as in all levels of government, are subject to revision over a period of time.

EXERCISES

1. How does the experimental confirmation process in the case of Newton's laws compare with the verification in the discovery of Neptune?
2. Criticize the passage by Newton, quoted above, from the standpoint of scientific method and scientific knowledge.

3. Investigate and criticize Newton's "Rules of Reasoning in Philosophy." See Newton's *Principia* (edition cited above), pp. 398–400. (Also, reading random passages from the *Principia* ought to prove rewarding.)

4. Give examples of "laws" from five of the following sciences (or divisions of knowledge) and attempt to classify them as either statistical generalizations, or postulates, or theorems: logic, mathematics, psychology, economics, sociology, biology, chemistry, physics.

5. Find and investigate a significant illustration of the scientific method involving the formulation of a law or principle, or comprehensive theory. Use a science of special interest to you. Suggested references are the following:

A Treasury of Science, edited by Harlow Shapley, Harper and Brothers, 1946, second edition.

The Development of the Sciences, edited by L. L. Woodruff, Yale University Press, 1941.

An Introduction to Reflective Thinking, Columbia Associates in Philosophy, Houghton Mifflin Co., 1923.

On Understanding Science, James B. Conant, Yale University Press, 1947.

5–5 Significance of the Einstein theory of relativity. One of the main influences in changing the attitude of physicists and astronomers toward Newton's laws of gravitation and the laws of motion has been the theory of relativity, developed by Einstein and other mathematical physicists during the first two decades of the 20th century. The theory is actually divided into two parts known as the *special theory* and the *general theory*. Many popular books, including some by Einstein himself, attempt to describe the two parts of the theory in an informal, nontechnical manner. Here we shall limit the discussion primarily to the significance of the Einstein theory for scientific method and scientific knowledge.

A. *The special theory.* The special theory of relativity was developed first, and the general theory came as a logical sequel and a generalized development of the principles of the other. The special theory had its origin in the kind of scientific perplexity or impasse with which we are now familiar, namely, a conflict between currently accepted theories or "laws" and certain observations and experiments.

The crucial difficulties arose in comparing *experiments* on the velocity of light with *theories* of the propagation of light. Here we omit many details, but the idea is essentially as follows. The conjecture had been made that space is not empty but is pervaded by

an "ether" (or "ocean") through which astronomical bodies move, and also through which light is propagated. According to this hypothesis, used in conjunction with Newton's laws of motion, light rays meeting a body moving through the ether would appear to have different velocities, depending on the direction of motion and the velocity of the body. Thus if a light ray is traveling through this hypothetical ether in the same direction that the earth is supposed to be traveling through it, an observer on the earth should find that the velocity of the light ray relative to him is smaller than it would be if the ray were traveling in the reverse direction.

For a rough analogy, suppose you are on a local train moving at 40 miles per hour, and an express train traveling at 60 miles per hour passes you, going in the *same* direction. The rate at which the express pulls away from you is then 20 miles an hour. On the other hand, if the express goes by in the *opposite* direction at 60 miles per hour, the rate of change of distance between your train and the express is 100 miles per hour.

Two physicists, Michelson and Morley, devised and carried out an experiment for the purpose of measuring the supposed variation in velocity of light rays traveling in different directions relative to the motion of the earth. The surprising results of this experiment were that the variation was nonexistent.

It will lend authority to our discussion here and frequently throughout the remainder of this section to quote from a valuable expository book, *The Evolution of Physics*, by Einstein and Infeld: *

> "The result (of the Michelson-Morley experiment) was a verdict of 'death' to the theory of a calm ether-sea through which all matter moves. No dependence of the speed of light upon direction could be found . . . Every experiment has given the same negative result as the Michelson-Morley one, and never revealed any dependence upon the direction of the motion of the earth."

Physicists were thus forced to discard as unsatisfactory the joint hypotheses of (a) the existence of an "ether-sea," and (b) the variable velocity of light depending on the direction of motion of the observer relative to the light ray. As an alternative to these discarded hy-

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* Albert Einstein and Leopold Infeld, *The Evolution of Physics*, Simon and Schuster, 1938, p. 183. Quoted by permission. Another helpful reference for this section is the following: L. Infeld, *Albert Einstein*, Scribner, 1950.

potheses, Einstein proposed the following simple assumptions as the foundation of his special theory:

(1) that the velocity of light is constant and remains entirely independent of the motion of the observer or of the source of light;

(2) that physical laws remain the same for all observers who are in uniform motion relative to each other.

The first of these assumptions corresponds to the known experimental facts. The second is of a more theoretical and general nature.

From these two fundamental postulates follow logically as theorems the more highly publicized and striking principles of the theory, such as the shrinking of moving measuring rods in the direction of their motion, and the differences in reading of clocks in uniform motion relative to each other. These theorems definitely conflict with Newton's basic ideas about absolute time and absolute motion, and also with his laws of motion. But the quantitative predictions based on Einstein's theory do not disagree significantly with those based on Newton's theory except for extremely high velocities (approaching the velocity of light, roughly 186,000 miles per second).

For a check of Einstein's postulates, experiments were conducted with radioactive elements yielding desirable high velocities. We quote again from Einstein and Infeld: *

"The experiments show that the resistance offered by these particles depends on the velocity, in the way foreseen by the theory of relativity. In many other cases, where the dependence of the resistance upon the velocity could be detected, there was complete agreement between theory and experiment. We see once more the essential features of creative work in science: prediction of certain facts by theory and their confirmation by experiment."

B. *The general theory.* In the general theory, Einstein aimed to develop a consistent logical system of broader scope than the special theory, which would include within it the principles of the special theory. At the same time he was attempting to give a more accurate and unified picture of the real physical world than classical physics had proved capable of doing. Einstein has pointed out that in achieving his purpose, he was actually attempting to replace a number of miscellaneous assumptions and laws by a few simple and

* *Op. cit.*, p. 207.

fundamental postulates. At the same time, this required the whole theory of the physical universe to become more highly abstract and mathematical. We quote again a significant passage from Einstein and Infeld: *

> "New difficulties arising in the development of science force our theory to become more and more abstract. But our final aim is always a better understanding of reality. Links are added to the chain of logic connecting theory and observation. To clear the way leading from theory to experiment of unnecessary and artificial assumptions, to embrace an ever-wider region of facts, we must make the chain longer and longer. The simpler and more fundamental our assumptions become, the more intricate is our mathematical tool of reasoning; the way from theory to observation becomes longer, more subtle, more complicated. Although it sounds paradoxical, we could say: Modern physics is simpler than the old physics and seems, therefore, more difficult and intricate."

This quotation could be used to offer strong support for the value of postulational organization in applied mathematics and science (see our previous discussion in Section 2–7).

One of the chief features of the general theory is that it replaces Newton's laws of gravitation by a geometric theory of gravitation. A fundamental assumption of this geometric theory is that physical space is not correctly described by the strict use of Euclidean geometry. (Newton had accepted Euclidean geometry without question as a valid picture of physical space.) In the general theory it is assumed that physical space is warped or distorted from the Euclidean variety by the presence of masses in motion. The warping hypothesis turns out to be logically sufficient to account for the Newtonian phenomenon of apparent force of attraction between masses. As a matter of fact, the general theory disagrees with predictions based on Newton's law of gravitation only for "strong" gravitational forces (much as the special theory contradicts the Newtonian laws of motion only for velocities which approach the velocity of light).

It is natural to ask whether this occasional disagreement with Newton's law of gravitation is supported by any experimental observations. The answer is "yes"; there are several observational situations which exhibit a closer degree of correspondence with pre-

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* *Op. cit.*, p. 225.

dictions based on the general theory than with predictions based on Newton's law.

We mention two of these situations: (1) the bending of light rays subject to strong gravitational action, and (2) the deviation of the planet Mercury from its orbit. In the first case Einstein predicted that during total eclipses of the sun there should be a bending of light rays coming from stars in the apparent vicinity of the sun. This prediction would not have been made on the basis of Newton's Law. Both the fact and amount of bending have been verified satisfactorily by astronomers in a number of different observations of eclipses.

In the second case, another logical consequence of the general theory was the prediction of a certain amount of discrepancy between the observed orbit of the planet Mercury and its theoretical orbit based on Newton's laws. A discrepancy had been previously known to astronomers, but unlike the case of the discrepancy in the orbit of Uranus, no satisfactory explanation had been found. Einstein's theory predicted the amount of the discrepancy satisfactorily.

These experimental confirmations of the general theory must be considered the more impressive because the theory was developed without any explicit prior concern about bending of light rays or deviation of Mercury from its orbit.

In spite of these successful predictions, Einstein hardly has any misconceptions as to the finality of his theory. He points out that any theory must continually meet the test of fitting the experimental observations, and he is supposed to have remarked once, "No amount of experimentation can ever prove me right; a single experiment may at any time prove me wrong."

Exercises

1. Compare, as far as you can, the viewpoints of Newton and Einstein toward scientific method and the nature of scientific knowledge, including such topics as degree of truth of fundamental principles, use of mathematics, and attitude toward confirmation. (This question might conceivably provide the basis for a doctor's thesis in the philosophy of science!)
2. Interpret from the standpoint of logic the statement attributed to Einstein, quoted above.
3. Investigate the status of Einstein's famous equation $E = mc^2$. How is it related to theory and observations?

5–6 Geometry in relation to physical science. It was pointed out in the preceding section that Einstein's general theory of relativity

relies heavily on a new geometric theory of physical space. From the standpoint of describing physical space, it is sometimes suggested that geometry can be studied as a branch of physics. Mathematicians usually frown upon this because by their definition geometry belongs to mathematics.

At the risk of confusing the terminology, let us consider the contrast between these two viewpoints — between geometry as a branch of physics and geometry as a branch of mathematics.

Geometry as a branch of mathematics is a collection of self-contained logical systems which can be developed on an abstract basis no matter how far removed they may seem from the real world. Geometry as a branch of physics, on the other hand, has as its main purpose the logical and accurate description of the observed facts of physical space. Geometry as a branch of mathematics contains numerous logical theories of highly imaginative types of "space." Geometry as a branch of physics is interested in those logical theories which might conceivably be useful in the process of describing observed or real space. Such geometric theories, according to our previous classification (in Chapters 1 and 2), may be considered as belonging to applied mathematics.

If someone asks: "*Which is the true geometry?*", the question is meaningless when we consider geometry as a branch of mathematics, whether pure or applied. As emphasized in Chapter 1, all we have a right to say in this case is: "*If* the postulates are true, *then* the theorems are true." On the other hand, the question is meaningful, or more nearly so, when we consider geometry as a branch of physics. In the former case we are primarily concerned that our postulates be logically consistent, that the undefined terms be explicit, and that the whole development from postulates and undefined terms be logically complete. In the latter case we are anxious to have a sound logical theory, but we want the *best* theory possible, i.e., a geometric system whose postulates and theorems agree as closely as possible with observations. In other words, we wish the postulates to stand the usual tests for hypotheses in *any* department of scientific or reflective thinking.

One of the main problems in geometry as a branch of physics, or let us say *physical geometry*, is the question of whether physical space is Euclidean or non-Euclidean. According to Einstein the answer would be "neither." The apparent paradox here is due to the fact that "non-Euclidean geometry," as we have seen, refers usually to

either one of two main varieties of geometry contradicting Euclid. These are the Lobachevskian and Riemannian geometries. The kind of geometry used by Einstein in his general theory of relativity is neither of these, nor is it Euclid's geometry. It is a four-dimensional geometry of "space-time," making good use of generalized differential methods in pure geometry, due to Riemann. The fact that Einstein's theory has been credited with a fairly high degree of experimental confirmation still does not prevent us from considering the narrower question of choice between the orthodox Euclidean and non-Euclidean geometries.

To be more specific, we might reword the question as follows: Which one of the Euclidean or non-Euclidean systems of geometry agrees most closely with the observed facts of physical space? Apparently we have to be content, at the present time, with an indeterminate answer. The crucial experimental test for choosing between these geometries might well be taken as the measurement of the sum of the angles of a physical triangle, say a triangle determined by three stars. So far no significant deviation from 180° has ever been discovered. (By "significant" deviation we mean statistical evidence of a sum which is consistently greater or less than 180° by an amount exceeding variations to be expected due to errors in measurement.)

Accordingly, we have no clear proof by the sum-of-the-angles test that one of the standard non-Euclidean geometries is preferable to ordinary Euclidean geometry. At the same time, we have no proof that Euclidean geometry is preferable to the others. For it may well be that there is actually a deviation from 180° in one of our large-scale astronomical triangles, but that our measuring instruments are not yet sufficiently precise to exhibit the deviation.

We can now understand the statement which is sometimes made to the effect that physical space might conceivably be proved to be *not* Euclidean, but it can never be proved to be definitely Euclidean.

5–7 Mathematical and scientific truth. The discussion of the role of geometry as a branch of mathematics and as a branch of physics can easily be generalized to yield a comparison of mathematical and scientific truth.

Our discussion indicates that scientific truth is in a different category from the abstract relative truth of pure or applied mathematics.

For it is the function of any science to formulate, with all possible aid from mathematics or logic, consistent logical theories which "fit" or explain observed facts as *closely and efficiently* as possible.

It is significant to note the word "efficiently." When a choice is to be made between possible rival theories the decision may be influenced not only by observational evidence, but also by the judgment of mathematicians or logicians concerning the relative simplicity and compactness of these theories. It will be recalled that Einstein advances arguments of this nature in support of his theory.

Scientific knowledge at any stage consists of the currently accepted scientific theories, together with known observational facts not yet explained by theoretical discussions. The latter unexplained facts, plus any contradictions arising between theories and observations, are always an important source of scientific problems, which in turn supply motivation for further scientific inquiry. New investigations lead to revision of previously accepted theories, and the revised theories yield new predictions, which again must be subjected to experimental testing. Thus the process repeats itself and there is no end in sight.

What then is the real nature of scientific truth in contrast to mathematical truth? The answer is suggested in a nutshell by another famous quotation from Einstein: "As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain they do not refer to reality." *

This statement indicates that there is a qualitative distinction which perhaps can be summarized in less striking language by describing mathematical truth as *abstract* and scientific truth as *empirical*. There is always this gulf between the two. But both varieties rely on logic and involve the kind of relativity inherent in any logical system of thought.

In addition, over a period of time, both mathematical truth and scientific truth are subject to revision. The former changes with changing standards of logical rigor within mathematics. The latter changes not only with this variable, but also with our changing capacity of observation of the real world, and with the changing theories proposed to describe that world.

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* Quoted by C. G. Hempel, "Geometry and Empirical Science," *American Mathematical Monthly*, Vol. 52, 1945, pp. 7–17.

Exercises

1. From the standpoint of advancement of scientific knowledge, would you say that mathematicians ought to devote their main research efforts to applied rather than to pure mathematics?

In answering this question it will be helpful to search for examples of mathematical theories which have been developed from the *pure* standpoint at one period of history, and applied to some other field at a later period. (Refer to Richardson, Chapter II, or Cooley, Gans, first edition, Chapter XXI, or second edition, Chapter 24, for suggestions. See also certain sections of the present text, e.g., Sections 1–7, 2–2, and 2–7.)

2. Interpret in your own words the above quotation from Einstein concerning the laws of mathematics and reality.

5–8 Conclusion. This completes Part I, in which we have attempted to make a general study of the postulational structure of logical thought and the active process of logical reasoning as used in the formulation of mathematical and scientific knowledge, together with applications of the logical process and postulational organization to other situations.

In Part II we shall continue with more extended illustrations of postulational systems in pure mathematics, and shall investigate, in some detail, modern postulational concepts and techniques.

PART II

SOME SIGNIFICANT POSTULATIONAL SYSTEMS, CONCEPTS, AND METHODS

CHAPTER 6

ELEMENTARY ALGEBRA AS A LOGICAL SYSTEM

6–1 Introduction. It is the object of this chapter to exhibit some fundamental portions of elementary algebra as a logical system.* In so doing we shall be following up the brief discussion of postulational organization of algebra in Chapter 2.

To the beginner, some of the most mysterious, and perhaps arbitrary, appearing rules of algebra are the rules for operating with positive and negative numbers. We shall show that the rules of signs, and some important rules concerning the numbers 0 and 1, can be proved as theorems in a development based on certain fundamental algebraic postulates, including most of those already mentioned in Chapter 2. Our approach will be incomplete — we will omit exponents, powers, radicals, quadratic equations, simultaneous equations, etc. — but at least it will be significant for the purpose of emphasizing logical relationships in elementary algebra which are often overlooked. At the same time, this approach should serve to introduce a more modern point of view both toward postulates and toward algebra as a whole.

6–2 The postulates of rational operations (Set A). In the study of arithmetic and high school or college algebra everyone gains practical but rather informal and intuitive acquaintance with different types of numbers, such as integers (i.e., whole numbers including 0 and 1), rational numbers (i.e., numbers expressible as the quotient of two integers), positive and negative numbers, and real and complex numbers, together with various operations and relations involving these numbers. A rigorous logical development of the whole complex number system usually is not attempted until an advanced stage of mathematical study — either the late college or early graduate school level — and even then the development is apt to be somewhat sketchy.

What we wish to do here is to formulate postulates corresponding

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* The present chapter is based in part on E. R. Stabler, "Demonstrative Algebra," *The Mathematics Teacher*, Vol. 39, 1946, pp. 255–260.

to what we believe, on the basis of our past experience, to be elementary properties of certain *sets* or *classes* of numbers. We shall have in mind especially properties of the class of real numbers, but we shall purposely avoid restricting ourselves to real numbers. This is because the postulates actually will hold good for numbers of more than one type. At the same time it will be useful to realize that the postulates do not hold for every possible set of numbers.

To emphasize the flexibility of interpretation of our postulates — and to avoid reliance on previous ideas about numbers except when we are specifically judging the truth or falsity of the postulates — we shall refer to "elements" rather than "numbers." It is equally desirable to be flexible in our interpretation of the operations used in our postulates, but we refer as usual to "addition" and "multiplication," with the understanding that we may wish to interpret these operations sometimes in nonstandard ways, for example, as "modulo addition" or "modulo multiplication."

Let us suppose, then, that we are given an *undefined* set or class of elements and two *undefined* operations, *plus* and *times*, for combining these elements. The undefined set of elements will be designated by the symbol K, while the individual elements of the set will be designated by such symbols as a, b, c . . . . (It is understood that distinct symbols such as a and b do not necessarily represent distinct elements. We may have $a = b$.) The undefined operations of addition and multiplication will be designated as usual.

From the standpoint of Chapter 4, the undefined words entering into the statement of the postulates make the postulates more like *propositional functions* than definite propositions.* And not until concrete meanings are assigned to the undefined terms can we say whether the postulates are true or false. If the postulates are true, we use the same terminology as for propositional functions, and say that they are "satisfied."

We shall give the postulates in two sets: (A) *postulates of rational operations* and (B) *postulates of signs and order*. Discussion of Set B will be postponed until after we have stated the theorems yielded by Set A.

In Set A all but one of the postulates are stated in pairs, referring to corresponding properties for addition and multiplication. It is

* We repeat here the definition of Sec. 4–2: A *propositional function* is a statement containing one or more variable terms which becomes a proposition as a result of substituting suitable meanings for the variable or variables.

understood in advance that $a + b$ is an element uniquely determined by a and b, and similarly for ab. We assume also, without explicitly listing them here, the familiar axioms of equality, and we shall not feel obliged to refer to them in proofs.

The postulates of Set A follow. The ordinary and primed numbers used in labeling the paired postulates refer to properties of addition and multiplication respectively.

A. Postulates of Rational Operations

Undefined terms: Set K of elements $a, b, c, \ldots$ combinable under two operations designated as *plus* and *times*, or *addition* and *multiplication*.

Postulates 1,1′. *Closure.* If a, b are any elements belonging to the set K, then $a + b$ and $a \cdot b$ are also elements belonging to K.

Postulates 2,2′. *Commutative operations.* If a, b are in K, then

$$a + b = b + a, \quad \text{and} \quad ab = ba.$$

Postulates 3,3′. *Associative operations.* If a, b, c are in K, then

$$(a + b) + c = a + (b + c), \quad \text{and} \quad (ab)c = a(bc).$$

Postulates 4,4′. *Identity elements.* There exist in the set K unique * *zero* and *unity* elements (designated as 0 and 1), distinct from each other, such that for every element a in K

$$a + 0 = a, \quad \text{and} \quad a \cdot 1 = a.$$

Postulates 5,5′. *Inverse elements.* For each element a in K there exists a unique corresponding element $\bar{a}$, and, except perhaps for 0, a unique corresponding element $/a$, such that

$$a + \bar{a} = 0 \text{ and } a \cdot /a = 1$$

($\bar{a}$ and $/a$ may be read respectively as "a bar" and "slant a.")

Definitions 1,1′. $\bar{a}$ is called the *opposite* (or *negative*) of a, and $/a$ is called the *reciprocal* of a.

Postulate 6. *Left distributive law.* If a, b, c are in K, then

$$a(b + c) = ab + ac.$$

Let us comment briefly on certain of the postulates. Postulates 1 and 1′, the closure postulates, actually are of great significance not

* By a "unique" element we mean one and only one element having the specified property.

only in this particular set of postulates but also in many other postulate sets for important algebraic systems. The role of closure is more readily appreciated by reference to a finite table of numbers such as the tables for adding days of the week or the tables of "modulo arithmetic" in Chapter 1. The closure property in these tables, it will be recalled, simply takes the form of using no other symbols in the body of the table than those appearing in the margins.

Postulates 2,2′, 3,3′, and 6 state the familiar commutative, associative, and left distributive properties for elements belonging to K. Note that we do not include the right distributive rule as a postulate here, the reason being that it is readily proved as a theorem from the commutative and left distributive postulates.

Postulates 4,4′ and 5,5′, the identity and inverse postulates, state familiar properties of 0 and 1, and of negatives and reciprocals of given elements.

Before proceeding to the theorems deducible from these postulates it would be desirable to make various concrete interpretations for the set K as a set of *numbers*, and to judge in which cases the postulates actually are all satisfied. Let us assume here that for all types of numbers the commutative, associative, and distributive rules are valid. (This assumption is justified by the advanced logical development of the number system, mentioned above and to be referred to again in the final section of the chapter.)

Example 1. Let K be interpreted as the set of all *positive integers*, and let $+$ and $\times$ have their usual meanings. The closure postulates are satisfied since the sum and product of two positive integers are also positive integers. Postulates 2,2′, 3,3′, and 6 are satisfied by our assumption. The number 1 is a positive integer and behaves as required by Postulate 4′, so this postulate is satisfied. But the number 0 is not included in the positive integers, and there is no positive integer which has the required property of a zero element. Hence Postulate 4 *fails*. Similarly, Postulates 5 and 5′ fail, as the required negatives and reciprocals of positive integers are not positive integers, with the exception of the reciprocal of the integer 1.

Example 2. Let K be the set of all *rational numbers* (positive, negative, and zero). Here all of the postulates in our list are satisfied. The sum and product of two rational numbers are rational numbers. The commutative, associative, and distributive postulates hold by our assumption. The rational numbers 0 and 1 serve

as identity elements. Every rational number has a negative (or *opposite*) which is also rational, and every rational number except 0 has a reciprocal which is rational.

EXERCISES

Determine for each of the following sets of numbers whether or not the postulates of Set A are satisfied. Tell why or why not, as explicitly as possible. (In the case of modulo systems the commutative, associative, and distributive postulates can be shown to hold always, as in ordinary arithmetic.)

1. The set of all positive and negative integers and zero.
2. The set of all real numbers, positive, negative, and zero.
3. The set of all rational numbers between $+1$ and -1.
4. The numbers 0, 1 taken modulo 2 (see Section 1–3).
5. The numbers 0, 1, 2 taken modulo 3.
6. The numbers 0, 1, 2, 3 taken modulo 4.
7. The numbers 0, 1, 2, 3, 4 taken modulo 5.

6–3 Theorems: Set A, Group 1. We now assume the abstract postulates of Set A and deduce theorems, still without committing ourselves to any definite set of elements. We claim simply that the theorems will be true for any set of elements for which all of the postulates are satisfied.

The development of theorems from the postulates of Set A will be given in three groups: *first*, those which follow from the unprimed postulates 1 to 5; *second*, those which follow from the corresponding primed postulates; and *third*, those which involve direct or indirect use of the distributive postulate.

The first group of theorems follows. Postulates will be referred to briefly in the proofs by names rather than by numbers.

THEOREM 1. $\bar{0} = 0$.

Proof.

$0 + 0 = 0$ by identity postulate.

Hence $\bar{0} = 0$ by inverse postulate

(since the inverse of any element of K is unique).

THEOREM 2. *If a is an element belonging to K, then*

$$\bar{\bar{a}} = a.$$

Proof.

$a + \bar{a} = 0$ by inverse postulate,

whence $\bar{a} + a = 0$ by commutative postulate,

and $\bar{\bar{a}} = a$ by inverse postulate.

The next theorem leads up to the operation of subtraction. The theorem says, in effect, that it is always possible to perform this operation within K. This is not trivial. For example, within the system of positive integers we know that subtraction is not always possible.

THEOREM 3. *If a, b are any two elements in K, there exists in K a unique solution of the equation $x + b = a$, namely, $x = a + \bar{b}$.*

Proof. I. *Suppose there is a solution x, in K; we show x is unique in accordance with the indicated formula.*

We have $x + b = a$ by hypothesis
and $\bar{b}$ is in K by inverse postulate.

Adding $\bar{b}$ to each side, we obtain

$$(x + b) + \bar{b} = a + \bar{b}.$$

Hence $x + (b + \bar{b}) = a + \bar{b}$ by associative postulate

(applied to left side of preceding step),

$x + 0 = a + \bar{b}$ by inverse postulate,
$x = a + \bar{b}$ by identity postulate.

Furthermore, $a + \bar{b}$ is in K by the closure postulate.

II. *We show by substituting $a + \bar{b}$ for x in the left side of the equation that $a + \bar{b}$ actually is a solution:*

$(a + \bar{b}) + b = a + (\bar{b} + b)$ by associative postulate,
$= a + (b + \bar{b})$ by commutative postulate,
$= a + 0$ by inverse postulate,
$= a$ by identity postulate.

Thus $a + \bar{b}$ is a solution.

In the preceding proof of Theorem 3 it should be observed that both parts of the proof are essential. The first part shows that if there is *any* solution (in K) it must be uniquely given by the stated formula. The second part corresponds to the familiar process of "checking" an answer to an equation in elementary algebra. (Incidentally, it is easy to find illustrations of equations in elementary algebra which apparently yield solutions under the assumption that there is a solution, and yet which on checking are found to have no solution after all. Or one apparent solution may be good and another "extraneous." For examples, see chapters on radical or fractional equations in algebra textbooks.)

We are now in a position to define officially the operation of subtraction within our set K.

DEFINITION 2. If c is the unique solution of the equation $x + b = a$, we say that c is the *difference*, *a minus b*, and write $c = a - b$.

THEOREM 4. *If a and b are any two elements of K, subtraction of b from a is always possible in K according to the rule* $a - b = a + \bar{b}$. (This theorem is simply a restatement combining Theorem 3 and Definition 2.)

Hereafter, when theorems or corollaries state equalities, it is to be understood that the hypothesis includes a clause like that of Theorem 4, such as "if a, b belong to K, ..."

COROLLARY 1. $a - \bar{b} = a + b$

(*for* $a - \bar{b} = a + \bar{\bar{b}}$ *which* $= a + b$ *by Theorem 2*).

COROLLARY 2. $a - b = 0$ *if and only if* $a = b$.

COROLLARY 3. $a - 0 = a$.

COROLLARY 4. $0 - a = \bar{a}$.

THEOREM 5. $b - a = (\overline{a - b})$.

Proof. It is sufficient by the inverse postulate to prove that

$$(a - b) + (b - a) = 0$$

(for this postulate says that the opposite of any number of K is unique). We prove this as follows:

$$\begin{aligned}
(a - b) + (b - a) &= (a + \bar{b}) + (b + \bar{a}) && \text{by Theorem 4,}\\
&= [(a + \bar{b}) + b] + \bar{a} \\
&= [a + (\bar{b} + b)] + \bar{a} && \text{by associative postulate,}\\
&= (a + 0) + \bar{a} && \text{by commutative and inverse postulates (or Theorem 2 and inverse postulate),}\\
&= a + \bar{a} && \text{by identity postulate,}\\
&= 0 && \text{by inverse postulate.}
\end{aligned}$$

Hence $b - a = (\overline{a - b})$ by inverse postulate.

THEOREM 6. $\bar{a} + \bar{b} = (\overline{a + b})$.

Proof. It is sufficient to prove that

$$(a + b) + (\overline{a + b}) = 0.$$

This is left as an exercise.

The following two theorems correspond to the usual rules of elementary algebra for removing parentheses preceded by + or − signs.

THEOREM 7. $a + (b - c) = (a + b) - c$ (or, for short, $a + b - c$).

THEOREM 8. $a - (b - c) = (a - b) + c$ (or, for short, $a - b + c$).

Proof.

$$\begin{aligned} a - (b - c) &= a + (\overline{b - c}) && \text{by Theorem 4,} \\ &= a + (c - b) && \text{by Theorem 5,} \\ &= a + (c + \bar{b}) && \text{by Theorem 4,} \\ &= a + (\bar{b} + c) && \text{by commutative postulate,} \\ &= (a + \bar{b}) + c && \text{by associative postulate,} \\ &= (a - b) + c && \text{by Theorem 4.} \end{aligned}$$

This completes the first group of theorems.

EXERCISES

1. Prove Corollaries 2, 3, 4 of Theorem 4.
2. Prove Theorem 7.
3. Does the associative principle hold in K for the operation of subtraction? Explain.
4. Prove $(a - b) + (c - d) = (a + c) - (b + d)$.
5. Prove $(a - b) - (c - d) = (a - b) + (d - c)$.
6. Prove $a - b = (a + k) - (b + k)$.

6–4 Theorems: Set A, Group 2. The second group of theorems might have been presented and proved in a column parallel to the first group, for the two groups correspond, theorem for theorem. Whereas addition is the fundamental operation involved in the first group, multiplication is fundamental in the second group. The first group depends entirely on the unprimed postulates, except Postulate 6, while the second group depends entirely on the primed postulates. With one exception the unprimed postulates 1 to 5 can be changed into the primed postulates 1′ to 5′ by changing addition to multiplication, so we have here a slightly imperfect "duality" (which, of course, does not extend to Postulate 6). The exception is that in Postulate 5′ no inverse or reciprocal is required for the element 0. Accordingly, any primed theorem which depends on Postulate 5′, directly or indirectly, will have a qualification in its statement. The form of this qualification will be the hypothesis that certain elements are not equal to 0.

It should be apparent that the proof of any primed theorem can

be given by use of the primed postulates and theorems with the same numbers as those used in the proof of the corresponding unprimed theorems. This is simply because, with the exception already noted, the unprimed postulates can be translated into primed postulates, so that the same translation can be made from unprimed to primed theorems. Logical relationships in the proofs of the primed theorems remain essentially unchanged. Accordingly, the proofs of the theorems of the second group will be omitted. It is advisable however, as an exercise, to work through some of these proofs in detail to see the actual process involved in "translating."

THEOREM 1′. $/1 = 1$.

THEOREM 2′. *If* a *is in* K, *and* $a \neq 0$, *then* $//a = a$.

THEOREM 3′. *If* a, b *are two numbers in* K, *with* $b \neq 0$, *there exists in* K *a unique solution of the equation* $x \cdot b = a$, *namely* $x = a \cdot /b$.

DEFINITION 2′. If c is the unique solution of the equation $x \cdot b = a$, (where $b \neq 0$) we say that c is the *quotient*, a *divided by* b, and write $c = a/b$.

THEOREM 4′. *If* a, b *are two numbers of* K, *with* $b \neq 0$, *division of* a *by* b *is always possible in* K *according to the rule* $a/b = a \cdot /b$.

COROLLARY 1′.	$a//b = a \cdot b$	*if* $b \neq 0$.
COROLLARY 2′.	$a/a = 1$	*if* $a \neq 0$.
COROLLARY 3′.	$a/1 = a$.	
COROLLARY 4′.	$1/a = /a$	*if* $a \neq 0$.
THEOREM 5′.	$b/a = /(a/b)$	*where* $a, b \neq 0$.
THEOREM 6′.	$/a \cdot /b = /(ab)$	*where* $a, b \neq 0$.
THEOREM 7′.	$a \cdot (b/c) = (ab)/c$	*where* $c \neq 0$.
THEOREM 8′.	$a/(b/c) = (a/b) \cdot c$	*where* $b, c \neq 0$.

EXERCISES

Prove:

1. $a/(b/c) = (ac)/b$, where $b, c \neq 0$.
2. $(a/b) \cdot (c/d) = (ac)/(bd)$, where $b, d \neq 0$.
3. $a/b = (ak)/(bk)$, where $b, k \neq 0$.
4. $(a/b)/(c/d) = (ad)/(bc)$, where $b, c, d \neq 0$.

6–5 Theorems: Set A, Group 3. We come now to the theorems which depend on the (left) distributive postulate. Without the dis-

tributive postulate we would be unable to prove certain key theorems concerning operations with the number 0. Due to the fact that there is no counterpart of the distributive postulate which can be obtained by interchanging multiplication and addition, we shall have no primed theorems in this group. First we note as a theorem the right distributive rule, the proof of which should be immediately obvious.

THEOREM 9. $(a + b)c = ac + bc$.

THEOREM 10. $a(b - c) = ab - ac$.

Proof. *Let* $d = b - c$. Then by Definition 2, we have

$$d + c = b,$$
$$ad + ac = ab \qquad \text{by distributive postulate,}$$
$$ad = ab - ac \qquad \text{by Definition 2,}$$

or $$a(b - c) = ab - ac.$$

COROLLARY. $$(a - b)c = ac - bc.$$

THEOREM 11. $a \cdot 0 = 0$.

Proof. Since $0 = b - b$, where b is an element in K, by Corollary 2 of Theorem 4,

$$a0 = a(b - b),$$
$$= ab - ab \qquad \text{by Theorem 10,}$$
$$= 0 \qquad \text{by Corollary 2 of Theorem 4.}$$

THEOREM 12. *Division of any nonzero element by* 0 *is impossible in* K, *and division of* 0 *by* 0 *is indeterminate.*

Proof. I. *Let* a *be different from* 0 ($a \neq 0$). To attempt division of a by 0 means, in accordance with the idea of Definition 2′, to *attempt* to solve the equation

$$0x = a.$$

But no matter what number x stands for (in K), we know by Theorem 11 that the left side of this equation is 0, whereas the right side is different from 0. Hence it is impossible to solve the equation in K.

II. *Let* $a = 0$. To divide a by 0 now means to solve the equation

$$0x = 0.$$

But *every* value of x (in K) satisfies this equation, by Theorem 11. Hence the solution is indeterminate.

It is worth noting that Theorem 12 is not sufficiently emphasized in elementary mathematical subjects, such as algebra and trigonom-

etry. Many misconceptions exist concerning "division by 0." But the simple fact is that in all systems of numbers satisfying our postulates division by 0 is impossible or indeterminate. (Sometimes the word "impossible" is used to cover both cases.)

Next we come to a related principle concerning the number 0 that is used correctly and with great frequency in algebra and related subjects.

THEOREM 13. *If* $ab = 0$, *where* a, b *are in* K, *then either* $a = 0$ *or* $b = 0$.

Proof. Either $a = 0$, in which case the stated conclusion holds, or $a \neq 0$. Suppose now that $a \neq 0$. Then, by the inverse postulate we know that $/a$ exists in K, and

$$(/a)(ab) = (/a)0 = 0 \quad \text{by Theorem 11.}$$

But
$$(/a)(ab) = [(/a)(a)]b \quad \text{by associative postulate,}$$
$$= 1 \cdot b \quad \text{by commutative and inverse postulates,}$$
$$= b \quad \text{by identity postulate.}$$

Hence $b = 0$, which completes the proof.

At this point we recall a definition used in connection with "modulo arithmetic" in Chapter 1, where we found that in certain systems it is possible to have Theorem 13 contradicted by the existence of "divisors of zero."

DEFINITION 3. If a and b are *nonzero* elements of K such that $ab = 0$, then a and b are called *divisors of zero.*

COROLLARY. *There are no divisors of zero in* K.

The remaining theorems of this section are important for the rules of signs in multiplication.

THEOREM 14. $a\bar{b} = \overline{ab}$.

Proof.
$$a\bar{b} = a(0 - b) \quad \text{by Corollary 4 of Theorem 4,}$$
$$= a0 - ab \quad \text{by Theorem 10,}$$
$$= 0 - ab \quad \text{by Theorem 11,}$$
$$= \overline{ab} \quad \text{by Corollary 4 of Theorem 4.}$$

COROLLARY. $\bar{a}b = \overline{ab}$.

THEOREM 15. $\bar{a}\bar{b} = ab$.

The proof of Theorem 15 is left as an exercise.

EXERCISES

1. Prove the corollary of Theorem 14, and prove Theorem 15.
2. Prove $\bar{a}/b = \overline{(a/b)}$.
3. Prove $\bar{a}/\bar{a} = 1$.
4. Prove $a/\bar{b} = \overline{(a/b)}$.
5. Prove $(a + b)/c = (a/c) + (b/c)$.
6. Prove $(a/b) + (c/d) = (ad + bc)/bd$.
7. Construct a chart showing the "logical ancestry" of Theorem 12 concerning the impossibility of division by zero. This should show as clearly as possible the previous theorems, the definitions, and the postulates used directly and indirectly in the proof of the given theorem. A final summary should be made, listing the postulates on which Theorem 12 ultimately depends.
8. Do the same for Theorem 15.

6–6 Postulates of signs and order (Set B). Although it may seem that Theorem 15 above corresponds to the usual rule of "minus times minus equals plus," actually we have not yet introduced any official distinction between "positive" and "negative" elements, or any concept of "greater than" or "less than." For this purpose we introduce the postulates of Set B as follows. Although the three new postulates of Set B appear to refer only to the concept of positiveness, they immediately permit introduction, by definition, of the concept of negativeness and the ordering relations ($<$ and $>$).

Postulates, definitions, and (in the next section) theorems will be numbered consecutively with those of Set A.

B. POSTULATES OF SIGNS AND ORDER

POSTULATE 7. *Law of trichotomy.* The set K contains a subclass P of positive elements such that for any element a belonging to K one and only one of the following statements holds:

(i) a *is positive* (i.e., a belongs to P).
(ii) $a = 0$.
(iii) $\bar{a}$ *is positive* (i.e., $\bar{a}$ belongs to P).

POSTULATES 8,8′. *Closure for positive elements.* If a and b are positive, then $a + b$ is positive, and ab is positive.

DEFINITION 4. If $\bar{a}$ is positive, we say that a is *negative.*

DEFINITION 5. Given any elements a, b, of K, if the difference, $a - b$, is positive, we say that a is *greater than* b $(a > b)$ and b is *less than* a $(b < a)$.

6-7 Theorems: Set B. It is understood now that the postulates of Set B as well as Set A are in force and, as usual, that a, b, c belong to K.

THEOREM 16. *If a is a positive element of K, $\bar{a}$ is negative.*

Proof. This follows from Definition 4, on recalling that $a = \bar{\bar{a}}$ (by Theorem 2).

THEOREM 17. *a is positive if and only if $a > 0$.*

THEOREM 18. *a is negative if and only if $a < 0$.*

THEOREM 19. *For any two numbers a, b of K, one and only one of the following holds:*

$$a > b; \quad a = b; \quad a < b.$$

Proof. This follows from Postulate 7, Corollary 2 of Theorem 4, and Theorem 5.

THEOREM 20. *Transitive property. If $a > b$ and $b > c$,* then $a > c$.

Proof. Since $a > b$, $a - b$ is positive by Definition 5. Since $b > c$, $b - c$ is positive by Definition 5.

$$\therefore \quad (a-b)+(b-c) \text{ is positive} \quad \text{by positive closure postulate.}$$

But $$(a-b)+(b-c) = a-c \quad \text{by theorems of Set A.}$$

$$\therefore \quad a-c \text{ is positive}$$

and $$a > c \quad \text{by Definition 6.}$$

THEOREM 21. *If $a > b$ and c is any element of K, then $a + c > b + c$.*

THEOREM 22. *If $a > b$ and c is positive, then $ac > bc$.*

Proof. Since $a > b$, $a - b$ is positive by Definition 5. Since c is given positive,

$$(a-b)c \text{ is positive} \quad \text{by positive closure postulate.}$$

But $$(a-b)c = ac - bc \quad \text{by Corollary of Theorem 10.}$$

$$\therefore \quad ac > bc \quad \text{by Definition 5.}$$

THEOREM 23. *If a, b are positive elements of K with $a \geqq b$ (i.e., a greater than or equal to b), then* *

(i) $a + (-b) =$ *the positive element,* $a - b$ (*or* 0 *if* $a = b$).

(ii) $(-a) + b =$ *the negative element,* $-(a - b)$, (*or* 0 *if* $a = b$).

(iii) $(-a) + (-b) =$ *the negative element,* $-(a + b)$.

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* Here we revert to the familiar use of the minus sign to designate the opposite of a given element.

THEOREM 24. *If a and b are positive elements of K, then*
(i) $a(-b) = (-a)b =$ *the negative element,* $-ab$.
(ii) $(-a)(-b) =$ *the positive element,* ab.

THEOREM 25. *The square of any nonzero element of K is positive.*

COROLLARY. *The unity element* 1 *is positive.*

This last corollary makes it easy to prove certain ordering relations for reciprocals of elements and hence relations concerning fractions. Some of these are suggested below as exercises.

This completes our sample postulational development of algebraic theorems.

EXERCISES

1. Prove Theorem 19 in detail.
2. Prove Theorem 21.
3. Prove Theorems 23 and 24.
4. Prove: If $a > b$, then $-a < -b$.
5. Prove: If $a > b$ and $c > d$, then $a + c > b + d$.
6. Prove: If $a > b$ and c is negative, then $ac < bc$.
7. Prove: (a) If $c > 0$, then $/c$ is positive. (b) If $c < 0$, then $/c$ is negative.
8. Prove: If $a > 1$, then $0 < /a < 1$.
9. Prove: If $0 < a < 1$, then $/a > 1$.
10. Prove: $a/b > c/d$ if and only if $ad > bc$, where $bd > 0$.

6–8 Comparison with the definitional approach to the number system. The advanced logical development of the number system, mentioned in Section 6–2, is essentially a definitional approach, whereas our development has been a sample of an abstract postulational approach.

In contrast with our approach, the definitional approach has the following characteristics: (1) it is a more concrete development; (2) it usually starts with assumed properties of the class of "natural numbers," or positive integers (1,2,3, . . .); (3) it builds up the number system by successive definitions of new types of numbers, together with their operations and ordering relations, and for each new set of numbers it proves theorems based on the definitions and on the properties of the previously defined sets of numbers. In this way, the properties of the sets of positive rational numbers, all rational

numbers, all real numbers, and all complex numbers can be developed.*

In the definitional approach the need for each new set of numbers defined is clearly shown by the fact that certain desired processes cannot be carried out in a more restricted set of numbers. For example, it is impossible to divide 5 by 2 in the system of positive integers, so it is desirable to introduce (positive) rational numbers to make this division possible. A positive rational number is defined as a pair of positive integers (written for example as m/n, with $n \neq 0$), and rules are defined for adding and multiplying with these new numbers. These defined rules are equivalent to theorems proved or provable in our development.

Again it is impossible to subtract 7 from 2 (more strictly 7/1 from 2/1) in the system of positive rationals, so negative rational numbers are introduced and operations for them defined so as to make subtraction always possible.

To show that irrational numbers are needed, the usual illustration is to attempt to solve the equation

$$x^2 = 2$$

within the system of rational numbers. Euclid gave a proof by indirect reasoning that such a solution is impossible, and this has become a standard example of an impossibility proof in mathematics. (For the proof, see Richardson, p. 108.) When irrational numbers and their operations are defined (by one of various standard methods, falling back on the system of all rationals), the enlarged system of numbers forms the system of all real numbers.

A simple process which cannot be performed within the set of real numbers is to solve the equation

$$x^2 + 1 = 0.$$

This leads finally to the definition of complex numbers and their operations.

The theorems proved in a definitional development of this kind show that our postulates of Set A — the "postulates of rational

* For an outline of this kind of approach, with details in the early stages, see Richardson, Chapters III and IV. For an alternative type of approach with more emphasis on complex numbers, see Marie J. Weiss, *Higher Algebra for Undergraduates*, John Wiley, 1949, Chapters 1 and 2. See also H. F. Fehr, *Secondary Mathematics*, D. C. Heath, 1951, Chapter 10.

operations" — are satisfied by the following sets of numbers with their accompanying operations: (1) the set of all rational numbers; (2) the set of all real numbers; (3) the set of all complex numbers. The rational numbers and the real numbers also satisfy the postulates of Set B — the "postulates of signs and order" — but the complex numbers do not.

Note. The reason why the complex numbers do not satisfy the postulates of signs and order can be appreciated if we recall (from trigonometric discussion of complex numbers) that every complex number has a square root which is a complex number, i.e., every complex number is the square of some complex number. Thus if it were possible to describe certain complex numbers as "positive" in such a way as to meet the requirements of the postulates of Set B, we would find in accordance with Theorem 25 that every nonzero complex number would be positive. On the other hand, any given nonzero complex number would be the opposite of some other (positive) complex number and thus would be negative as well as positive. This would contradict Postulate 7, the law of trichotomy.

For an exhaustive postulational treatment of the algebra of complex numbers see the monograph, "The Fundamental Propositions of Algebra" by E. V. Huntington, in *Monographs on Topics of Modern Mathematics,* edited by J. W. A. Young; Longmans, Green, 1927. This monograph will be discussed in Section 11–5.

It may be inquired which type of approach is preferable — the abstract postulational approach or the more concrete definitional approach. Probably the answer is that both are needed. Actually, the two approaches constitute logical developments in reverse directions, and are comparable to large-scale necessary or sufficient conditions. Thus in order to have a reasonable basis for defining operations with new types of numbers, it is *proved*, in the postulational approach, what the definitions *necessarily* must be if we wish any system of numbers to have the standard properties specified in the postulates. Then in the definitional development it is shown, conversely, that the definitions dictated by the postulational approach are *sufficient* to guarantee that the desired properties hold. The analogy should not be pushed too far, however. There are other considerations which make the postulational study desirable, as will appear in Chapters 8 and 9.

CHAPTER 7

AN ABSTRACT GEOMETRIC SYSTEM

7–1 Introduction. In the preceding chapter we attempted to show the value of a postulational approach in algebra. The results achieved were familiar, but the logical organization and the rather abstract point of view were presumably novel.

In the present brief chapter we shall give a sample of an abstract, postulational approach to geometric subject matter which will appear novel in contrast with both Euclidean and non-Euclidean geometries. The peculiar feature of the geometric system to be developed is that it contains only a finite number of points and lines. This characteristic makes it very convenient for study as an example of a concise logical system of postulates and theorems. At the same time it makes it easy to give a rigorous treatment, and thus avoid Euclid's defects.

Both Chapter 6 and the present chapter will be useful to us as a basis for more detailed discussion of postulational concepts and methods in Chapter 8.

7–2 Postulates for a finite geometry. The system to be developed will be called a "finite" or "miniature" geometry.* We shall be concerned with a class S of undefined objects called "points" and "lines," with the qualification that a "line" is simply a collection of "points." We must be careful not to read any more meaning into these terms than is justified by the postulates. In particular, we must not be prejudiced by our previous experience with diagrams and ordinary geometric principles. The "geometry" will turn out to be of an unusual and restricted type.

The postulates are as follows. (In the postulates and theorems, if two or more points or lines are mentioned it will be understood always that we mean *distinct* points or lines.)

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* For somewhat more elaborate examples of miniature geometries see Richardson, Chapter XVII; Cooley, Gans, second edition, Chapter 24; or John W. Young, *Fundamental Concepts of Algebra and Geometry*, Macmillan, 1925, p. 41 ff.

POSTULATE 1. Each pair of lines in S have at least one point in common.

POSTULATE 2. Each pair of lines in S have not more than one point in common.

POSTULATE 3. Every point in S is on at least two lines.

POSTULATE 4. Every point in S is on not more than two lines.

POSTULATE 5. The total number of lines in S is four.

As in Chapter 6, we note the analogy of abstract postulates to propositional functions. This would become still more noticeable if we replaced "point" and "line" respectively by the symbols x and y. Then the first postulate would read, "each pair of y's in S have at least one x in common," and so on for the other postulates. Or "point" and "line" might be replaced equally well by nonsensical words such as "punk" and "lunk." Either of these procedures would dramatize the absence of truth or falsity in the postulates as they stand. As in the case of propositional functions, we want to know whether the postulates are *satisfied or not* when appropriate meanings are assigned to the variable (undefined) terms.

7–3 Consistency of the postulates. One question that naturally arises is, can we really have *all* of these postulates satisfied at the same time? Or is it possible that there are some internal contradictions among them? If we can find some interpretation of the class S of points and lines for which all of the postulates become true, then we ought to be convinced that they are logically consistent.

For this purpose, let us interpret class S as an orchard of trees arranged in specified rows, with "point" interpreted as *tree* and "line" as *row of trees.* Then we shall want four rows in S, with each possible pair of rows having one and only one tree in common, and also with each tree belonging to two and only two rows. This requires that the number of possible trees in S be the same as the number of distinct pairs of things (rows) which can be chosen from four things. Tabulation, or the theory of combinations, tells us that the number of such pairs is six. Thus we want to have exactly six trees, arranged in our four rows. If we count each tree the number of times it occurs in different rows we get twelve tree-counts to be distributed among four rows. Thus to lay out our orchard we presumably want to have our six trees arranged in four rows with three trees in each

row. This is easily accomplished as shown in Fig. 7–1, where the trees are represented by A, B, C, etc., and the rows by the dotted paths I, II, III, IV.

The rows can be specified as follows:

Row I. $A, B, C.$
Row II. $A, D, E.$
Row III. $B, F, E.$
Row IV. $C, F, D.$

Note that there is really no question of whether or not the rows are "straight," and this has been deliberately suggested by curving the rows in the diagram. Also, the rows are not like line segments; on the contrary, they are entirely empty between the trees. It is simply a matter of assigning trees to designated rows in such a way as to satisfy the specifications of the postulates. (We do not permit the arbitrary creation of new rows in addition to those officially listed.) A quick check shows that the suggested arrangement of trees in rows actually meets all the specifications. Accordingly, we can justifiably claim that our arrangement of trees constitutes a proof of logical consistency for the postulates.

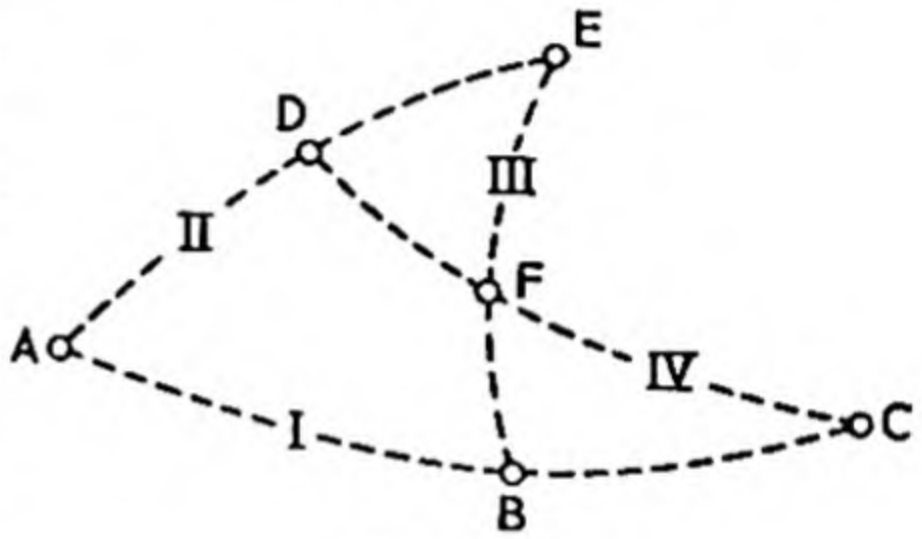

Fig. 7–1

An alternate interpretation of the undefined terms that is still further removed from ordinary geometric concepts can easily be suggested. For this interpretation, let the class S of "points" and "lines" be the class of members and committees of a club having exactly six members (A, B, C, D, E, and F) and exactly four committees (I, II, III, IV). The membership of each committee is as follows:

Committee I. $A, B, C.$
Committee II. $A, D, E.$
Committee III. $B, F, E.$
Committee IV. $C, F, D.$

It is apparent that this example is essentially the same as the example of the orchard. The only difference is that "trees" are changed to "club members," and "rows" to "committees." Accordingly, the

club interpretation is equally good for convincing us of the consistency of our postulates.

EXERCISES

1. Translate the abstract postulates of our finite geometry into statements about members and committees of a club, using the interpretations of "point" and "line" suggested above.

2. Verify that the committee lists suggested above for a club of six members actually satisfy the specifications of all translated postulates obtained in Exercise 1.

7–4 Theorems of the finite geometry. In formulating an interpretation of our postulates as a consistency proof, we have already, in effect, used some logical consequences of the postulates. We proceed to formulate these and other results as abstract theorems. The theorems will then, of course, hold good for any interpretation which satisfies all of the postulates. For generality, we avoid specifying what interpretation we have in mind.

THEOREM 1. *Every pair of lines in S have one and only one point in common.*

Proof. This follows at once from combining Postulates 1 and 2.

THEOREM 2. *Every point in S belongs to two and only two lines.*

Proof. This follows by combining Postulates 3 and 4.

THEOREM 3. *There are exactly six points in S.*

Proof. By Postulate 5 there are exactly four lines in S; call them I, II, III, IV.

By Theorem 1 each pair of these lines have exactly one point of S in common. Call the point common to lines I and II: I–II, etc. Then we have the following points belonging to S:

I–II, I–III, I–IV, II–III, II–IV, III–IV.

By Theorem 2 every point belonging to S must be contained in this list. Likewise, by Theorem 2, no two of the listed points can coincide, for in that case we would have a point belonging to more than two lines.

THEOREM 4. *There are exactly three points on each line in S.*

Proof. This follows immediately from the labeling of the six points of S, in Theorem 3. The labeling indicates the precise pair of lines

on which each point lies. Thus on line I, we have the points

I–II, I–III, I–IV,

and no others. Similarly, on line II, we have

I–II, II–III, II–IV,

and no others; and similarly for lines III and IV.

DEFINITION. Two points which have no line in common are called *parallel* points.

THEOREM 5. *Each point in S has exactly one point in S parallel to it.*

Proof. A given point of S is determined by (or lies on) exactly two lines of S. Thus there is exactly one point, determined by the remaining two lines, of S which has no line in common with the given point.

For example, if we are given the point I–II, the parallel point is III–IV.

COROLLARY. *On a line not containing a given point of S there is one and only one point of S which is parallel to the given point.*

This corollary suggests an unusual transformation of the Euclidean parallel postulate. The corollary is obtainable from the latter by the process of interchanging the words "point" and "line," and making slight adjustments in terminology. It could be called the geometric *dual* of the Euclidean parallel postulate. This reminds us of dual properties of propositions and classes (discussed in Chapters 3 and 4) and dual or almost dual properties of ordinary algebra (in Chapter 6).

Duality between *point* and *line* is a significant concept in the two-dimensional version of a very general branch of geometry known as projective geometry. For example, in projective geometry two points always determine a line and two lines determine a point. Every theorem in projective geometry can be changed into a dual theorem which is equally good. Figures can be dualized. For example, a *complete quadrilateral* is defined as the figure consisting of four lines, no three through a common point, and the six points of intersection of these lines taken in pairs. (This is analogous to our orchard of trees and rows, except that the "lines" of projective geometry are thought of as made up continuously of points, as in ordinary

Euclidean geometry.) The dual figure of the complete quadrilateral is a *complete quadrangle*, defined as the figure consisting of four points, no three of which are on a common line, and the six lines joining these points in pairs.

Duality, whether logical, algebraic, or geometric in background, is a useful concept as an aid to understanding the importance of abstract form rather than concrete substance, both in the active process of logical reasoning and in the completed structure of a logical system. For this reason, in the exercises below we suggest the task of dualizing the miniature geometry we have been developing in this chapter.

EXERCISES

1. Suppose a club operating under appropriately translated versions of Postulates 1 through 5 wishes to take in two new members (without dropping any old ones). (a) What should be done to solve the constitutional difficulties? (Assume that it is desired to keep Postulates 3, 4, and 5 in force and that the least possible modification is to be made in the other postulates.) (b) What effect would the changes proposed in (a) have on the theorems?

2. Dualize the entire system of abstract postulates and theorems of our finite geometry to form a new finite geometry. Is it strictly necessary to give a separate consistency proof for the new postulates, and separate proofs for the new theorems?

3. Investigate the Desargues Triangle Theorem of projective geometry. This has the unusual property that its converse is identical with its dual. (See, for example, R. Courant and H. Robbins, *What is Mathematics?*, Oxford University Press, 1941, Chapter IV.)

7–5 Significance of the finite geometry. It may be objected that the logical system developed in this chapter is of such restricted application that it is trivial. From the standpoint of any probable usefulness in connection with the laying out of orchards or the organization of clubs, this objection is certainly justifiable! From the standpoint of usefulness as an illustration of postulational systems, concepts, and methods, however, quite the contrary is the case.

From ancient times down to the present, mathematicians have always concerned themselves with many problems and theories which appear to be remote from the real world. For example, we have noted in Chapter 1 that the ancient Greek geometers spent much time trying to perform certain construction problems with straightedge and compasses. They did not succeed, and we know now that they could not have succeeded, but in spite of this (as

emphasized in Section 2–2), their attempts were far from useless because they did develop useful mathematical concepts and theories as by-products of their efforts. Again, certain hypothetical gambling problems, under investigation by the mathematicians Fermat and Pascal in the early 17th century, led to the beginning of the mathematical theory of probability. Probability theory today is fundamental in the applied mathematical theories of life insurance and statistics. Many similar examples could be cited. The point is that problems or theories in the brain-teaser category often show their usefulness by stimulating more extended and profitable mathematical thought, or by furnishing neat and clarifying examples of the general nature of mathematical thought.

It is the latter advantage that we claim for our miniature geometry, for it illustrates in a limited amount of space the contrast between Euclid's brand and the modern brand of postulational organization. The special features of the modern approach as illustrated here can be summarized as follows:

(1) Undefined terms are officially agreed upon in advance of the postulates.

(2) The statements made by the postulates are abstract, *neither true nor false*, in their original form and thus are in the nature of propositional functions rather than propositions.

(3) The undefined terms can be given a variety of interpretations for which the postulates can be judged to be true or false or, in alternative language, satisfied or not satisfied.

(4) A proposed set of postulates is considered to be logically sound, or consistent, if some concrete interpretation can be found for which all of the postulates are judged to be satisfied simultaneously.

(5) A given set of postulates may be satisfied by quite different types of concrete interpretations.

(6) The theorems which follow from the postulates will hold good in every interpretation for which the postulates all hold good.

(7) The system of postulates and theorems can be viewed either as an abstract logical structure or, with a specified interpretation, as a concrete theory concerned with the indicated subject matter.

In the next chapter, we shall continue the study of consistency of postulates and introduce some new postulational concepts.

CHAPTER 8

POSTULATIONAL CONCEPTS AND METHODS. FIELDS

8–1 Introduction. We now make use of the two preceding chapters to give a detailed discussion of certain important characteristics which sets of postulates may possess, namely, consistency, independence, completeness, and categoricalness; and we explore available methods for determining whether or not a given set of postulates has these characteristics.

In the case of consistency, much of our discussion will be amplification of what has already been suggested in Chapter 7. As illustrative material and as a basis for the discussion of important algebraic systems known as *fields*, we shall make considerable use of the postulates introduced in Chapter 6.

8–2 Postulational inconsistency and consistency. The primary qualification for an acceptable set of postulates from the mathematical point of view is, of course, logical consistency. Although the meaning of consistency should be reasonably clear by now, we shall try to sharpen the concept by the following pair of formal definitions, in which we first define inconsistency. This may seem roundabout, but it appears to be the most satisfactory approach.

DEFINITION 1. A set of postulates is *logically inconsistent* if the joint assumption of the postulates, for every possible interpretation of the undefined terms, leads by logical deduction to a contradiction of the form "q is true *and* q is false."

DEFINITION 2. A set of postulates is *logically consistent* if the set is not inconsistent.

Combining the two definitions, we see that a set of postulates is logically consistent if and only if it is *not* the case that, for *every* concrete interpretation of the undefined terms, the joint assumption of the postulates leads to a contradiction. If we produce a concrete interpretation for which all of the postulates are judged to be true, then we claim that the postulates in this case cannot yield a contradiction (assuming the validity of the law of contradiction for the

concrete interpretation in question) and hence that the postulates are indeed logically consistent. This gives us the method of establishing consistency which is already familiar to us.

Note. The logical background of this standard consistency method can be explained on a somewhat oversimplified basis by use of quantified propositions as discussed in Chapter 4.

Let P_v stand for a list of postulates expressed in the form of propositional functions which use certain undefined or variable words, all signified by the subscript v. For each collective interpretation of v, P_v becomes the conjunction of the given postulates, now in the form of definite propositions. If we claim that we have found an interpretation for which all of the postulates are satisfied we are claiming that

$$(\exists v)(P_v),$$

or that

$$(\exists v)\{\sim[P_v \rightarrow q(\sim q)]\},$$

assuming that the law of contradiction holds for the specified interpretation. Now using rule Q-3, we obtain

$$\sim\{(v)[P_v \rightarrow q(\sim q)]\},$$

which, in turn, says it is false that for every interpretation the postulates lead to a contradiction. This justifies the assertion that the postulates are consistent.

An interpretation satisfying all of the postulates is sometimes referred to as a *model* of the postulates. If the specified concrete meanings of the undefined terms are substituted in the theorems deduced abstractly from the postulates, of course the theorems will also be judged to be true for the model. The entire logical system of postulates and theorems then can be said to be transformed from an *abstract theory* into a *concrete theory*. The word *model* can also be used to apply to the relation of the concrete theory to the abstract theory.

We may seem to have a watertight method for proving consistency whenever it is possible to exhibit the required model of the postulates. But actually it is far from an absolute proof. For example, we are assuming that the interpretation, for which the postulates are all judged to be satisfied, is itself free from contradiction (or, as stated above, that the law of contradiction applies to the interpretation), and that the process of judging has itself been correctly carried out.

In the case of the suggested models of the postulates for our miniature geometry, the judging was of a simple nature and could be carried out by direct observation. In other cases, however, we shall

find that the judging required is of a more complicated type and may itself involve long chains of reasoning. Furthermore, the assumed self-consistency of the interpretation often is equivalent to assuming the consistency of some other set of postulates, thus shifting the burden of the consistency proof. Logicians and research specialists in the foundations of mathematics have still to find any absolute type of consistency proof.

EXERCISES

1. Show by actual deduction that the following set of abstract logical premises, (a) to (d), is inconsistent (for all propositions p, q, r):

(a) p is true if and only if q is false.
(b) q is a sufficient condition for the falsity of r.
(c) r is true.
(d) p is false.

2. In ordinary algebra, what is meant by consistency or inconsistency of a set of simultaneous equations? Compare with consistency or inconsistency of a set of postulates.

8–3 Consistency of the "postulates of rational operations." Fields. In Chapter 6 we presented the "postulates of rational operations" as the basis for a sample logical development of elementary algebra. We made a point of using an abstract form of statement which would not ordinarily be used in elementary algebra — for example, referring to "elements" rather than "numbers." From the standpoint of flexibility in postulational interpretation, we can appreciate better now why it is desirable to do this.

The abstract terminology is also desirable from the standpoint of showing relationships with the wide branch of mathematics known as *modern algebra*, or *abstract algebra*. This is because modern algebra, in contrast with traditional algebra, is very much concerned with systems of *elements* (not necessarily numbers) whose properties are abstractly specified by sets of postulates. The result is that modern algebra is more rich and varied in subject matter, and at the same time tends to be more logically coherent in structure, than traditional algebra. Traditional algebra is still important, of course, but it is no longer the center of attention in the domain of algebra, just as Euclidean geometry is no longer the main interest in the realm of geometry.

We now show the connection of the postulates of rational operations with modern algebra by introducing the concept of a *field*.

DEFINITION. Any set of elements, with appropriate operations of addition and multiplication, for which all of the "postulates of rational operations" (Set A) of Section 6–2 are satisfied is called a *field*.

This definition is a good illustration of a common (and at times confusing!) mathematical habit of giving simple, everyday names to rather complicated ideas. Two other important types of systems in modern algebra, namely, *groups* and *rings*, illustrate the same tendency. (These will be discussed in Chapter 9.)

For reference, we summarize the postulates of rational operations here as field postulates.

POSTULATES FOR A FIELD

Undefined terms: Set K of elements $a, b, c, \ldots$ combinable under the two operations of addition and multiplication.

POSTULATES:

Closure postulates for addition and multiplication.
Commutative postulates for addition and multiplication.
Associative postulates for addition and multiplication.
Left distributive postulate.
Identity postulates for addition and multiplication.
Inverse postulate for addition.
Inverse postulate for multiplication (with no reciprocal required for zero).

If we are now asked for a consistency proof for the postulates of a field, we can use as an interpretation any one of the various sets of numbers which in Chapter 6 were shown to satisfy the postulates of rational operations. This assumes that the properties of these numbers have been adequately established ahead of time, presumably by the concrete definitional approach mentioned in that chapter. For a consistency proof there are disadvantages in some of these interpretations, of the nature suggested at the end of the preceding section.

The simplest model of a field is the modulo 2 system, which is already very familiar to us and which consists of the two elements 0, 1, with the following operation tables:

+	0	1
0	0	1
1	1	0

×	0	1
0	0	0
1	0	1

Without anything more complicated than direct checking of all the cases, we can show that the closure, commutative, associative, identity, inverse, and left distributive postulates all hold. (This should have been the result of Exercise 4 in Section 6–2.)

Accordingly, we should have no doubt of the consistency of the postulates for a field. Furthermore, we see that a wide variety of different types of systems form fields. For example, the number of elements in a field may be finite or infinite, and a field may or may not satisfy the "postulates of signs and order" (Set B of Chapter 6). We can speak of the field of all rational numbers, the field of all real numbers, and the field of all complex numbers. In spite of the varied nature of these different systems, we know that for every field the theorems of Set A of Chapter 6 hold good.

8–4 Fields and modulo systems. Finite fields (i.e., fields consisting of a finite number of elements) are useful for the purpose of novel illustrations. Among these are certain modulo systems.

We recall that a modulo m system (where m is a positive integer) contains the number symbols $0, 1, 2, \ldots, m-1$, and addition and multiplication tables are formed by ordinary arithmetic, except that results are recorded as remainders (themselves one of the symbols $0, 1, 2, \ldots, m-1$) above a multiple of m.

Given a modulo m system, we can test as follows to see whether the various field postulates are satisfied.

Closure postulates. These hold by the method of constructing the addition and multiplication tables.

Commutative, *associative*, and *distributive* postulates. These hold as in ordinary arithmetic. For complete details here it would be necessary to fall back on some elementary principles of the theory of numbers.

Identity postulates. 0 and 1 are always in any modulo system, as required.

Inverse postulate for addition. For each element in a modulo system there is always an opposite element, as required. Thus in the modulo 5 system we have $2 + 3 = 0$, so the opposite of 2 is 3 ($\overline{2} = 3$) in this system. In the modulo m system, if k is one of the elements (i.e., k is an integer such that $0 \leqq k < m$), then the opposite of k will be simply the integer $m - k$ (or 0 if $k = 0$).

Inverse postulate for multiplication. This postulate will be satisfied if and only if each nonzero element a in the given modulo system

has a reciprocal ($/a$) in the system or, in other words, if and only if we can find within the system a solution of the equation $ax = 1$.

This is clearly satisfied for the modulo 3 system, since here $1 \cdot 1 = 1$ and $2 \cdot 2 = 1$. In the case of the modulo 5 system the postulate is also satisfied, because each of the nonzero elements (1, 2, 3, 4) has the required reciprocal in accordance with the modulo 5 equalities:

$$1 \cdot 1 = 1, \quad 2 \cdot 3 = 1, \quad 3 \cdot 2 = 1, \quad 4 \cdot 4 = 1.$$

In the case of the modulo 4 system, on the other hand, we find that there is no reciprocal for the number 2, inasmuch as

$$2 \cdot 1 = 2, \quad 2 \cdot 2 = 0, \quad 2 \cdot 3 = 2$$

in this system.

We are now able to state that *a given modulo system will or will not be a field according as the inverse postulate for multiplication is or is not satisfied.* Accordingly, the modulo 3 system and modulo 5 system are fields, while the modulo 4 system is not a field. After working the exercises below, it should be easy to state a general rule concerning the modulus m which will decide whether or not a modulo m system is a field.

Exercises

1. Do the following sets of numbers, with the usual operations, form fields? If not, tell which postulates fail to be satisfied. (a) The set of all positive integers. (b) The set of all positive and negative integers and 0. (c) The set of all rational numbers. (d) The set of all irrational numbers.
2. List the opposite and reciprocal of each element of the modulo 5 field.
3. Is the modulo 7 system a field? If so, list the reciprocal of each element of the system.
4. Is the modulo 6 system a field? Why or why not?
5. If a modulo system contains "divisors of zero," can it be a field?
6. A positive integer m is called *composite* if it can be written as the product of two positive integers, each of which is less than m. Otherwise, if greater than 1, it is called *prime* (in which case the only positive integral factors of m are itself and 1). With this definition in mind, state and prove a *necessary condition* (concerning the modulus m) for a modulo m system to be a field.
7. A famous theorem in theory of numbers, due to the French mathematician Fermat in 1679, makes the following assertion:

If p is a prime number and a is any integer which is not a multiple of p, then

$$a^{p-1} = 1 + \textit{some multiple of } p.$$

This is usually written

$$a^{p-1} \equiv 1 \pmod{p},$$

and read "a^{p-1} is congruent to 1 modulo p."

To show the significance of this theorem in a modulo p system, suppose a is any symbol $\neq 0$ in such a system. Then we can write

$$a^2 \equiv b \pmod{p},$$

where b is one of the remainders $1, 2, \ldots, p-1$ over a multiple of p. This can be rewritten

$$\begin{aligned} a^2 &= b + \text{multiple of } p, \\ \therefore \quad a^3 &= ab + \text{multiple of } p \\ &= c + \text{multiple of } p, \end{aligned}$$

where c is one of the remainders $1, 2, \ldots, p-1$ (so that in the modulo p table we read $ab = c$). Continuing in this way, we reach

$$a^{p-2} = k + \text{multiple of } p,$$

where k is either $1, 2, \ldots$ or $p-1$, and finally,

$$\begin{aligned} a^{p-1} &= ak + \text{multiple of } p \\ &= 1 + \text{multiple of } p, \text{ by Fermat's Theorem.} \end{aligned}$$

In the modulo table we then read $ak = 1$.

What does this result prove concerning a modulo p system, and how can the answer be correlated with Exercise 6?

8–5 Ordered fields. Idempotent fields. In this section we introduce, for the purpose of illustration, two special types of fields to be characterized by additional postulates.

One of the noticeable differences between the field of real numbers and the field of complex numbers is that the former satisfies the postulates of signs and order (Set B) of Chapter 6, while the latter does not. To emphasize the difference, we make the following definition:

DEFINITION. Any field which satisfies the "postulates of signs and order" (i.e., the trichotomy postulate and the closure postulates for positive elements) is called an *ordered field*.

We repeat here the special postulates for an ordered field. We presuppose a subclass P of positive elements within the class K of elements of the field.

TRICHOTOMY POSTULATE. For every element a belonging to K, one and only one of the following statements holds: a is positive, $a = 0$, or $\bar{a}$ is positive.

CLOSURE POSTULATES FOR POSITIVE ELEMENTS. If a and b are positive elements of K, then $a + b$ is a positive element, and ab is a positive element.

Both the field of rational numbers and the field of real numbers are ordered fields. If we grant the consistency of either the set of all rational numbers, or the set of all real numbers — and the consistency of these depends ultimately on the consistency of the natural numbers (1, 2, 3, . . .) — we have an interpretation establishing the consistency of the combined postulates for an ordered field.

We note that from the postulational standpoint the definition of an ordered field simply concentrates our attention on an enlarged set of postulates obtained by adjoining the postulates of Set B to those of Set A. The act of enlarging the original list of postulates has the effect of narrowing the type of interpretation which will satisfy all the postulates, and at the same time, of course, increasing the list of theorems. Thus in any ordered field we have the theorems of Set B available, and any such field agrees abstractly in these respects with the systems of all rational numbers or all real numbers.

In contrast with these latter systems, we have seen that certain modulo m systems form finite fields. Exercises 6 and 7 of the preceding section should have led to the following result:

THEOREM. *A modulo m system is a field if and only if m is a prime integer.*

Could any of these finite fields be at the same time ordered fields?

To answer the question, let us use as an example the modulo 3 system which we shall designate as the field F_3. If F_3 could be an ordered field, we know by a previous corollary (of Set B) that 1 would have to be a positive element. But

$$\bar{1} = 2$$

and

$$2 = 1 + 1,$$

which is positive by the positive closure postulate. Thus we would have both 1 and $\bar{1}$ positive, contradicting the trichotomy postulate. Hence F_3 could not be an ordered field.

The proof that other modulo systems which are fields cannot be ordered fields is essentially the same. Or we can use a slight variation of the proof just given by noting that in a modulo p field, F_p (where p is a prime integer), the sum of p terms each equal to 1, must

yield 0, while the sum of a smaller number of terms yields one of the elements 2, 3, . . . , $p-1$. If F_p were an ordered field, we would then have the result that 0 is a positive element, again contradicting the trichotomy postulate.

The second special type of field to be mentioned in this section is indicated by the following postulate and definition.

IDEMPOTENT POSTULATE. For every element a in K,

$$a \cdot a = a.$$

DEFINITION. An *idempotent field* is a field for which the idempotent postulate is satisfied.

If someone objects that the idempotent postulate is inconsistent with the other field postulates, of course the modulo 2 system, F_2, refutes the charge. For we know this system meets the conditions for an idempotent field. (We have only two elements, 0 and 1, in the field, and each of these elements is idempotent, i.e., $0 \cdot 0 = 0$ and $1 \cdot 1 = 1$.)

Of course, none of the other special fields we have been talking about — such as the fields of rational numbers, real numbers, complex numbers, and likewise all modulo p fields, where p is a prime other than 2 — are idempotent fields.

Modulo p systems, in general, satisfy a somewhat generalized idempotent postulate to the effect that for every element a in K:

$$a^p = a.$$

This follows from Fermat's Theorem in number theory (see Exercise 7 of the preceding section).

EXERCISES

Explain whether or not the following proposed sets of postulates are consistent:

1. The field postulates *and* the statement, "it is possible to divide an element a of K by 0 and get a unique quotient."
2. The field postulates *and* the statement, "for every a in K, $a + a = 0$."
3. The field postulates *and* the statement, "for every a in K, $a + a + a = 0$."
4. The field postulates *and* the statement, "there exist divisors of zero in K."
5. (a) The field postulates *and* the statement, "there is a denumerable infinity of elements in K." (b) The field postulates *and* the statement,

"there is a *nondenumerable* infinity of elements in K." (See the Exercises at the end of Chapter 1.)

6. The field postulates *and* the statement, "$a + b = a - b$ for every pair of elements a, b in K."

7. The field postulates *and* the statement, "for every a, b, in K, $(a + b)^p = a + b$, where p is a prime integer."

8-6 Independence of postulates. In listing the postulates for a field, suppose someone adds to Set A the right distributive postulate,

$$(a + b)c = ac + bc.$$

Is it incorrect to include this postulate in the list?

If we refer to the development of theorems from Set A in Chapter 6 we find that the right distributive rule appears there as Theorem 9. Thus the rule is a correct property of any field, but since it is provable as a theorem it should not be added to the list of postulates. Our whole plan of developing a certain branch or sub-branch of mathematics lies in logical organization, i.e., the postulate-theorem structure. This requires by its very nature a discrimination between assumed and proved statements. So, from the theoretical point of view at least, any statement labeled as a postulate which is capable of being proved from certain of the other postulates should be so proved and relabeled as a theorem. If retained as a postulate it is logically superfluous or *redundant.*

On the other hand, suppose it is suggested that we incorporate with Set A a new postulate:

d: There exist *divisors of zero* in K.

We know that the *negation* of d is a theorem following from Set A, so that the enlarged set would now be inconsistent. The situation seems quite different from the illustration concerning the right distributive postulate, but there is this similarity: in each case, the postulates of Set A enable us to *prove* or *disprove* the proposed new postulate. In neither case do we have any freedom of choice about accepting or rejecting the properties suggested in the new postulates, assuming always that the postulates of Set A are in force.

As an illustration of a property for which we do have freedom of choice, consider again the idempotent postulate in relation to the postulates of Set A (the field postulates). We know that the idempotent postulate *is consistent* with the field postulates, inasmuch as it does hold in at least one concrete illustration of a field — for

example, in the modulo 2 system. On the other hand, in most fields (for example, the modulo 3 system, where $0 \cdot 0 = 0$, $1 \cdot 1 = 1$, but $2 \cdot 2 = 1$, not 2) the idempotent postulate does not hold. Accordingly, it must be that neither the idempotent postulate, nor its negation, is logically deducible from the field postulates. In other words, the idempotent postulate is neither provable nor disprovable in the abstract logical system based on the postulates of Set A.

Now we make the following definitions:

DEFINITION 1. A postulate (or statement) q is said to be *independent* of certain other postulates, designated collectively as P, if neither q nor the negation of q is logically deducible from P.

DEFINITION 2. An entire set of postulates is said to be an *independent set* if each postulate is individually independent of the remaining postulates of the set.

Using the new terminology, we can make the following statements:

(1) The right distributive postulate *is not independent* of the postulates of Set A.

(2) The postulate d ("there exist divisors of zero in K") *is not independent* of the postulates of Set A.

(3) The idempotent postulate *is independent* of the postulates of Set A.

Testing for independence. The basis of our claim in statement (3) above suggests the method to be used in testing an entire set of postulates for independence.

First, we make sure that the given set is consistent by the usual interpretation method. (If not consistent we presumably go no further.) Consistency guarantees that it cannot be that the negation of any one postulate of the given set is deducible from the remaining postulates of the set (for then the entire set of postulates would always lead to a contradiction).

Next, we try to find an interpretation of the basic undefined terms for which Postulate 1 of the given set *fails* while each of the remaining postulates *is satisfied*. If we are successful, this shows that Postulate 1 is not redundant in the given set — i.e., it cannot be a logical consequence of the remaining postulates. (For if it were a logical consequence, then it would have to be true in *every* interpretation for which the other postulates are true.) In the same way, we try to show that Postulate 2 is not redundant by producing a new concrete interpretation for which Postulate 2 fails while each of the other

postulates holds good, and so on to the end of the list of postulates. If we are successful in each case, we can then claim that the entire set is independent.

A complete proof of independence for a set of a dozen or more postulates will probably be a painstaking process. Usually such a proof is not considered mathematically essential, even in advanced mathematical research, except in postulational investigations. It is, of course, considered mathematically desirable to avoid any obvious redundancies among the postulates. But from the psychological or pedagogical point of view it is often thought justifiable to start a mathematical development with a definitely redundant set of postulates.

Returning now to our postulates for a field (Set A), it seems quite likely that the set is independent. It is possible to provide independence proofs for many of the postulates as they stand, but in other cases some revision of statement would be needed in order to implement the desired proofs.

In the case of the five consistent postulates for our finite geometry of Chapter 7, it is not difficult to supply independence proofs for each postulate. For convenience, we restate the postulates here:

POSTULATE 1. Each pair of lines in S have at least one point in common.

POSTULATE 2. Each pair of lines in S have not more than one point in common.

POSTULATE 3. Every point in S is on at least two lines.

POSTULATE 4. Every point in S is on not more than two lines.

POSTULATE 5. The total number of lines in S is four.

As an independence example for Postulate 1, let us interpret S to consist of the points A, B, C, D, E distributed among four lines I, II, III, IV as follows:

I. A, B.
II. A, C, D.
III. B, D, E.
IV. C, E.

In this interpretation Postulate 1 fails, since the pair of lines I, IV have no point in common. On the other hand, no pair of lines have more than one point in common, so Postulate 2 is satisfied. Each of the points A, B, C, D, E is on at least two lines and not more than

two lines, so Postulates 3 and 4 are satisfied. We have four lines in S, so Postulate 5 is satisfied. Hence Postulate 1 is independent of the other postulates. In the same way, independence proofs can be supplied for Postulates 2, 3, 4, 5.

The concept of independence and the method of testing for independence can be applied equally well in the case of definitions or theorem-hypotheses involving various conditions which are jointly listed. Consider the following conceivable (but not conventional) definition for a square in Euclidean plane geometry:

> A square is a figure which meets the following conditions: (i) it is a quadrilateral; (ii) its sides are all equal; (iii) its angles are all equal.

The conditions (i), (ii), (iii) are consistent in Euclidean geometry, as can be shown by constructing a figure which satisfies all three conditions. Furthermore, no one of the conditions is redundant, as shown by the following examples:

Example 1. Consider a figure which is an equilateral triangle. (i) fails but (ii) and (iii) are satisfied.

Example 2. Consider a rectangle whose length is twice its width. (ii) fails but (i) and (iii) are satisfied.

Example 3. Consider a rhombus containing angles of 60° and 120°. (iii) fails but (i) and (ii) are satisfied.

Thus the definition as stated meets the test for independence of the specified conditions (i) to (iii).

It is worth while emphasizing this application to definitions. A common examination error made by both high school and college students of mathematics is to include redundant statements in definitions. For example, what should we say about the following purported definition of "parallelogram"?

> A parallelogram is a quadrilateral whose pairs of opposite sides are parallel and equal.

Clearly a student of geometry who gives this as a definition shows a lack of logical discrimination, inasmuch as the condition of *equal* opposite sides can be proved by use of the condition of *parallel* sides. In fact, the former is invariably stated as a theorem, and the standard definition is restricted to the latter. Aside from lack of logical discrimination, definitions which contain obvious redundancies are unnecessarily cumbersome in use and require an unnecessary amount of checking to find out whether they are satisfied.

Exercises

1. Supply independence examples for Postulates 2, 3, 4, 5 of the finite geometry postulates. (*Hint:* For Postulates 2 and 3, S can be taken to consist of *seven* points. For Postulates 4 and 5, just *one* point in S is sufficient, but there are more interesting examples which use *four* points for Postulate 4 and *three* points for Postulate 5.)

2. Are the following statements, taken separately, independent of the field postulates?

(a) $a0 = 0$.
(b) If $ab = 0$, then either $a = 0$ or $b = 0$.
(c) The number of elements in K is finite.
(d) The number of elements in K is infinite.
(e) For every a in K, $a + a + a = 0$.
(f) For every a in K it is possible to divide a by 0 and get a unique quotient.

3. Criticize the following set of postulates given by a student on an examination as a purported list of postulates for our finite geometry:

(a) Every point in S is on at least two lines.
(b) No point in S is on more than two lines.
(c) There are four lines in S.
(d) No three lines in S have a common point.

4. Criticize the following statement of a hypothesis in Euclidean geometry: "Given an equilateral triangle ABC such that angle $A = B = C = 60°$."

8–7 Weakness of postulates. Interdeducibility. Closely related to the concept of independence for postulates (or definitions or hypotheses) is the concept of *weakness* (and strength). The idea is essentially this: given two statements s and t, we say that t is weaker than s (and s is stronger than t) if t is a logical consequence of s but s is not a logical consequence of t. Familiar abstract examples would be the following:

p *or* q (i.e., $p \vee q$) is weaker than p;
p is weaker than p *and* q (i.e., pq).

For a concrete example, our identity postulates for fields called for the existence of a *unique* zero element and a *unique* unity element. It would be possible to weaken these postulates by dropping out the word "unique." This does not mean that we are not interested in the uniqueness property, but rather that it is superfluous to assume it in the postulates because uniqueness can be proved as a theorem by the aid of other postulates. To remove such excess conditions is

desirable from the standpoint of assuming as little as possible and proving as much as possible. Also, it is often an advantage in the process of testing postulates for consistency or independence (or in just checking to see whether the postulates are satisfied by a given interpretation) to have each postulate as weak as possible.

If we should weaken the identity postulates for a field as just described and likewise delete the word "unique" in the inverse postulates, we would change Set A into a new set of postulates Set A′. Taking both sets of postulates jointly, we could show that Set A′ is deducible from Set A, and Set A from Set A′. That is, the two sets are *interdeducible*, and thus logically equivalent. Hence, the abstract theory of fields would come out the same way no matter which set of postulates was used as a starting point.

From the technical standpoint Set A′, the weaker set, would be considered preferable. But from the psychological standpoint of an introduction to the theory of fields, Set A may well be better. As in the case of independence of postulates, weakness is a refinement which often is not insisted upon.

The concept of interdeducibility of postulate sets, briefly illustrated here, is important for the purpose of identifying equivalent approaches to the same abstract mathematical theory.*

8–8 Completeness of postulates. We have seen that all fields agree in the properties expressed by the postulates and theorems of Set A. But they show variety in other ways. For example, they may or may not be infinite, they may or may not be ordered, or they may or may not be idempotent.

For reasons such as these, we say that the set of postulates for a field is *not complete*. The word "complete" is used with somewhat varying meanings by different writers, but here we shall define it as follows:

DEFINITION. A given set of postulates is said to be *complete* if it is impossible to formulate any new postulate, expressed in terms of the given undefined concepts, which is independent of the given postulates.

To show specifically that the field postulates are not complete it is

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* This concept is illustrated at further length in Chapter 9, when we exhibit the interdeducibility of two different postulational foundations for groups, and also for Boolean algebras.

sufficient to cite the example of the idempotent postulate, which we have seen is independent of the postulates of Set A.

If a set of postulates is complete, the definition indicates that any new postulate, proposed for possible incorporation with the original set and expressed in terms of the original undefined terms, must fail to be independent of the original set. This failure can happen, of course, in two ways (as already illustrated in Section 8–6 in connection with adding certain postulates to Set A): by having either the new postulate or its negation logically deducible from the given set of postulates.

We can state the following rather surprising result of the definition of completeness:

THEOREM. *Any inconsistent set of postulates is complete.*

Proof. Given an inconsistent set of postulates designated collectively as P, it must be that, for every interpretation of the undefined terms,

P is false.

By using the paradoxical principle of implication that a "false proposition implies any proposition," [principle (a) of Section 3–4] we can now say that, for every interpretation, any proposition can be deduced from the assumption of P. In particular, any proposed new postulate q (*and also* the negation of q) can be deduced. Thus the given set is complete.

Naturally, we shall be more interested in the possible completeness of a consistent set of postulates than in the known completeness of an inconsistent set. Completeness is hardly to be considered a redeeming virtue for inconsistency!

If a set of postulates is both *consistent* and *complete*, we can say that all controversial questions which might arise within the abstract deductive system governed by the given postulates (or any model thereof) are *theoretically* decidable one way or the other (but not both!). This is because any statement formulated as a new postulate will either be provable or disprovable in the system (because of completeness), and cannot be both provable and disprovable (because of consistency). This is far from saying, however, that proofs or disproofs of particular statements are automatically made available.

In contrast, if a set of postulates is *consistent and incomplete* there will be certain statements, formulated in terms of the given undefined

concepts, which remain independent of the given set and which accordingly are neither provable nor disprovable in the system — as we have seen in the case of the field postulates.

The whole problem of completeness is usually considered on a highly technical basis in connection with a formalized theory of deductive procedure, i.e., a theory of "correct" methods of making proofs. From this point of view, fundamental research, especially by K. Gödel dating from about 1930, has shown that it is exceptional to find a set of postulates which is both consistent and complete. In fact, the combination of consistency and completeness appears to be unattainable for most branches of mathematics.

However, there is a property related to completeness which often is attainable together with consistency. This is categoricalness, which we discuss in the next section.

Before proceeding to that section we note that the topics under discussion here, and throughout much of this chapter, belong to a department of knowledge which is concerned with mathematics and logic, and yet remains outside these subjects. It has been referred to by some as *metamathematics*, by others as the *methodology* or *semantics of deductive systems*. For further light on these matters we refer to an elementary text by an outstanding contemporary logician and *metamathematician*, Tarski.*

8–9 Categoricalness of postulates. Isomorphism. We have used (in Section 8–5) an example of an idempotent field of two elements. Would it be possible to have an idempotent field of more than two elements? The answer to this question and some related considerations afford a convenient introduction to the concepts of this section.

Let us assume that we are given an arbitrary idempotent field, say F_1. Then, in accordance with the idempotent postulate, we must have for every element a in F_1

$$a^2 = a.$$

Hence $$a^2 - a = 0$$

and $$a(a - 1) = 0,$$ by standard theorems for any field.

Since there are no divisors of zero in a field, we must have, as in elementary algebra,

$$a = 0 \quad or \quad a = 1.$$

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* A. Tarski, *Introduction to Logic*, Oxford University Press, 1946, Chapter VI.

Thus every element of F_I is either 0 or 1, and since these are distinct elements in any field, we have proved the following theorem.

THEOREM. *In an idempotent field there are exactly two elements, namely, the zero element and the unity element.*

This theorem tells us that our previous illustration of an idempotent field — the modulo 2 system, F_2 — is typical of all idempotent fields with regard to number of elements. Is it also typical in other respects?

To investigate this, let us indicate the zero and unity elements of our arbitrary idempotent field, F_I, by z and u to emphasize abstractness. Let us formulate the operation tables for F_I.

We know by the identity and commutative postulates for addition that

$$z + z = z$$

and

$$z + u = u + z = u.$$

Likewise, we must have

$$u + u = z,$$

for if both

$$u + u = u \quad \text{and} \quad u + z = u,$$

we would obtain $u = z$ by adding the opposite of u to each side in each equation. Similarly, we obtain

$$u \cdot u = u,$$

and from the standard theorems about multiplying by 0,

$$z \cdot u = u \cdot z = z \cdot z = z.$$

Thus the operation tables can be summarized as follows:

$+$	z	u
z	z	u
u	u	z

$\times$	z	u
z	z	z
u	z	u

We see at once that these tables are precisely a translation of the modulo 2 tables, when 0 is translated as z and 1 is translated as u. Of course, the modulo 2 tables can be obtained equally well from these by translating in the reverse direction. In these circumstances, we say that the modulo 2 field, F_2, and the abstract idempotent field, F_I, are *isomorphic*, in accordance with the following definition.

DEFINITION 1. Given two sets of elements, E, E', each considered with respect to certain operations (the same *number* of operations

in each case). If it is possible to arrange a complete one-to-one correspondence between the elements of E and E' which holds throughout corresponding operation tables of E and E', then the two sets are said to be *isomorphic*.

This is often summarized by saying that the elements of two isomorphic systems can be put in one-to-one correspondence completely (paired) in such a way as "to preserve operations." All that this means is the following: If an operation of E is performed on elements a, b of E to yield the answer c and the corresponding operation is performed on the corresponding elements a', b' of E', then the answer in E' is the element c' which has *already* been paired with c in the given one-to-one correspondence.

A convenient illustration can be obtained by thinking of the multiplication table up to 10×10 written out correctly first in English and then in French. The two tables are certainly isomorphic under the pairing of English and French words for the same numbers. A multiplication problem given in one language can be translated to another language, if desired, and performed there, and the answer must be the translation of the answer to the given problem in the first language.

A similar situation prevails if we think of multiplication tables for integers written with different number bases. The multiplication table up to "10×10" using base ten is isomorphic to the multiplication table up to "13×13" using base seven.

The above definition of isomorphism can be broadened to provide for sets of elements considered also with respect to certain relations, for example, the ordering relation, $<$, which is available in ordered fields. In such a case, the pairing would require that whenever $a < b$ holds in set E, then the corresponding relation $a' < b'$ holds in set E'.

Now we are in a position to define the main concept of this section, *categoricalness*.

DEFINITION 2. Given a set of abstract postulates involving elements, and operations or relations. If any two interpretations satisfying all of the postulates are isomorphic, then the given set of postulates is said to be *categorical*.

We can now say that the set of postulates for an idempotent field is categorical. This is because we have shown that any interpretation satisfying all the postulates (taken abstractly as the field F_I)

must be isomorphic with the modulo 2 field. Hence any two models of the postulates must themselves be isomorphic. This is also shown by the fact that any two models must have operation tables which are obtainable from or translatable into the abstract tables for F_I by the translation process suggested. This leads us to the conclusion that with reference to properties which can be expressed in terms of the undefined concepts of an abstract field, all idempotent fields are abstractly in entire agreement with each other.

This is very different from the case of arbitrary fields, which sometimes contain a finite and sometimes an infinite number of elements, so that the required one-to-one correspondence between the elements of two arbitrary models could not possibly be set up.

The concept of categoricalness, as defined above, applies especially to postulates of an algebraic type. For use with postulates of other types (e.g., geometric) modified definitions of categoricalness would be required. In some cases, such a definition might not be very feasible.

If a set of postulates is known to be categorical we can say that the postulates are *sufficient* to determine an abstract uniformity of properties for all models of the postulates. This is because all models of the postulates are isomorphic. Hence any proposed statement formulated in terms of the original undefined concepts either (a) is satisfied in all models, or (b) fails in all models. In this respect, categoricalness agrees with completeness. But in a categorical set it may still not be possible to *prove* or *disprove* such a statement abstractly by formal rules of deductive proof. This is why we cannot claim that a categorical set has to be complete.

On the other hand, if the isomorphism concept is definable for a certain set of postulates, and the set is known to be noncategorical, then the set will also be incomplete.

Finally, we note that a categorical set is not always preferable to a noncategorical set. The categorical postulates for an idempotent field certainly are of very limited usefulness in comparison with the noncategorical postulates for an arbitrary field. When we develop the theory of idempotent fields we simply study abstractly the properties of the modulo 2 system. When we develop the theory of arbitrary fields, on the other hand, we study the properties common to many different systems — including all modulo p systems, and also the fields of all rational numbers, all real numbers, and all complex numbers. Thus a mathematical theory based on a noncategorical

set of postulates permits a unified logical treatment of the common part of differing logical systems. In contrast, a categorical set of postulates must lead to what is essentially a single, clearly delimited mathematical theory.

In the latter sense there is considerable significance in seeking a categorical set of postulates for the real number system or the complex number system, or a categorical set of postulates for Euclidean geometry. If such a set is produced, then the postulates can be looked upon as containing the theory of the corresponding branch of mathematics *in a nutshell*, i.e., with no key principles overlooked. In Chapter 11 we shall refer to certain well-known formulations of categorical postulate sets of this kind.

Before closing this chapter we shall discuss some applications of postulational methods to science and to nonmathematical situations.

EXERCISES

1. If each of the following postulates is taken separately in combination with the field postulates, will the resulting postulate sets be categorical? Why or why not?

(a) There is a finite number of elements in K.

(b) There are exactly two elements in K.

(c) There are exactly three elements in K.

(d) K satisfies the additional postulates for an ordered field.

2. Prove that the modulo 2 system and the odd-even system (odd + even = odd, etc.) are isomorphic systems.

3. Show that the addition table for integers 0, 1, 2, 3 taken modulo 4 is isomorphic to the multiplication table of integers 1, 2, 3, 4 taken modulo 5.

4. The concept of isomorphism for models of our finite geometry postulates can be defined as follows: Two models S_1, S_2 are isomorphic if we can establish a one-to-one correspondence between the points and lines of S_1 and the points and lines of S_2 such that whenever a point belongs to a line in S_1 the corresponding point belongs to the corresponding line in S_2.

Prove that the postulates for our finite geometry are categorical.

Suggestion. Show that for an arbitrary interpretation S satisfying all of the postulates, we can label the points and lines of S in the following two ways which seem to be different:

I. A, B, C		I. A, B, C
II. A, D, E	or	II. A, D, E
III. B, D, F		III. B, E, F
IV. C, E, F		IV. C, D, F

But the second method of labeling can be changed into the first by inter-

changing the labels D and E. Therefore any two models could have their points and lines labeled in accordance with the first interpretation, which would thus establish the desired isomorphism.

8–10 Postulational concepts in science. We have seen that the pattern of logical structure familiar in mathematics is an important guide in the formulation of scientific theories. Hence it is natural to look for useful applications of postulational concepts in the area of science. The most significant applications are those of consistency and independence.

Consistency is an all-important criterion for any set of postulates in science, as in mathematics or any other logically developed subject matter.

Methods of investigating the consistency of postulates in science cannot be expected to be the same as in mathematics. For one thing, the fundamental assumptions of a branch of science do not depend on such flexible undefined terms as those with which we deal in mathematics. Typical basic concepts in a scientific theory certainly are abstract to a greater or less degree, but all are meant to have their counterparts in the real world and are hardly subject to interpretation in the same manner as undefined *elements* and *operations* in an abstract algebraic theory. Accordingly, the attempt to ensure consistency of scientific postulates usually boils down to an attempt to obtain harmony between theory and observations, both on the level of postulates and of theorems.

Mere logical consistency of postulates, of course, is not sufficient to satisfy scientists. Unlike mathematicians, scientists require consistency of their postulates *with* experimental facts, and we recall that such consistency is commonly tested indirectly by comparing the logical consequences of the postulates with observed data. In case the comparison shows agreement (within the limits of experimental error), there is *confirmation* of the postulates, indicating that the consistency of the postulates with the experimental facts is plausible — at any rate, not disproved. Disagreement with observational data results in *infirmation* of the postulates, indicating definite disproof of the required type of consistency between abstract theory and concrete reality. A process ending with infirmation is often useful as a stimulus to revision of previously accepted scientific theories, and thus serves as an important aid to the evolution of scientific knowledge.

Occasionally a closer analogy with mathematical consistency methods may be possible. A model or interpretation may be useful to support the logical consistency of a proposed set of scientific assumptions. For example, the physical postulates underlying the structure of an *electron* might be argued to be consistent by using as a model the orbitary structure of planets in the solar system. Also, a favorite device of Einstein in supporting some of his fundamental assumptions has been to imagine abstract or "idealized experiments" dealing with such familiar situations as falling elevators or moving trains. Mental experiments of this type are in conformity with the spirit, if not the letter, of testing a set of mathematical postulates for consistency by the technique of concrete interpretations or models.

Other things being equal, the choice between two rival scientific theories would be decided in favor of that theory which seems to stand up better under the test of internal logical consistency and simplified logical structure. This is well illustrated by the Einstein theory again. Consider the following significant passage from Einstein and Infeld:*

> "The relativity theory arose from necessity, from serious and deep contradictions in the old theory, from which there seemed no escape. The strength of the new theory lies in the consistency and simplicity with which it solves all these difficulties, using only a few very convincing assumptions."

This same quotation can be used to illustrate the scientific significance of *independence* of postulates. The quotation claims that the Einstein theory employs "only a few . . . assumptions." This is certainly an advantage in any scientific theory. In Chapter 5 we mentioned the two fundamental assumptions of Einstein's special theory, and we noted that these assumptions lead logically to certain striking theorems, such as the shrinking of moving measuring rods. Also the famous mass-energy equation,

$$E = mc^2,$$

appears as a theorem. Einstein might conceivably have used principles of this type as additional or alternative assumptions, but it strengthened his position to minimize the number of assumptions and keep them as simple as possible, and to rely as much as possible on logical deduction.

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* A. Einstein and L. Infeld, *The Evolution of Physics*, Simon and Schuster, 1938, p. 203.

From the postulational standpoint, the attempt to discard superfluous postulates from a scientific theory may be considered an attempt to approach the ideal of an independent set of postulates. But it is really more than this. If a set of assumptions is found to involve logical redundancies, surely some scientific progress is achieved. The progress here is largely by way of unification and simplification of the logical structure of the theory. At the same time, a redundant postulate, newly recognized as such, may provide a previously unsuspected basis for experimental confirmation or infirmation of the remaining postulates.

To conclude this section, we mention a famous example of logical unification in physics, accomplished by one of Einstein's predecessors, Clerk Maxwell (1831–1879). Here we quote from a valuable essay by Margenau on the historical development of physics.* The reference to "all known experimental facts" means the facts of electromagnetism:

> "... The physicist is not content with facts. He desires to see them embedded in a consistent framework of mathematics not to be found in the field of observations.... Starting with four simple differential equations, whose elegance and symmetry are still the delight of mathematicians, [Maxwell] was able to deduce from them all known experimental facts; moreover, he was led to announce an entirely new and startling result: the existence of electromagnetic waves which were later shown to be identical with radio and light waves. Physical theory was becoming more and more comprehensive, drawing into its range an ever-increasing diversity of observable facts."

Exercises

1. Review the inconsistencies which led to the formulation of Einstein's Special Theory.

2. Investigate the postulational relationship of Kepler's laws of planetary motion and Newton's laws of gravitation and motion. (See E. T. Bell, *The Development of Mathematics*, McGraw-Hill, 1940, pp. 144–148.) What was the significance of this relationship for astronomical science?

3. In what way might scientific progress result if Mendel's laws of heredity, in biology, were found to be logical consequences of the theory of evolution?

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* See H. Margenau, "Physics," Chapter III in *The Development of the Sciences*, edited by L. L. Woodruff, Yale University Press, 1941, pp. 108–109.

4. How did Galileo defend the consistency of the theory that the moon revolves about the earth at the same time that the earth revolves about the sun? (Refer to "Proof that the Earth Moves," by Galileo, in *A Treasury of Science*, second edition, edited by Harlow Shapley, Harper, 1946, pp. 58–61.)

5. A pioneering attempt to apply postulational methods, and also symbolic logic and formalized deduction, in biology has been undertaken by the English scientist J. H. Woodger. Investigate an account of this in T. A. Goudge, "Science and Symbolic Logic," *Scripta Mathematica*, Vol. 9, 1943, pp. 69–80. (Or for the original source, see J. H. Woodger, *The Axiomatic Method in Biology*, Cambridge University Press, 1937.)

8–11 Postulational concepts in nonmathematical situations. In Chapter 2 we emphasized the importance of postulates in any logical organization of thought, whatever the subject matter might be. At that point we were more concerned with the logical development from the postulates used as premises than with analysis of the postulates themselves. In this section we suggest some applications of the standard postulational concepts in nonmathematical situations.

As always, any set of postulates, or statements jointly asserted, should be consistent in order to be acceptable.* Numerous familiar examples can be supplied.

A witness testifying in a court trial becomes discredited if there are contradictions in his testimony. A debater suffers a severe blow if his opponent can show that his fundamental premises are inconsistent. An executive in business or any branch of government soon has difficulties with his associates if his policies exhibit inconsistencies (especially in the form of prejudicial treatment of other persons).

Returning to the judicial sphere, one of the most important duties of the United States Supreme Court is to decide on the constitutionality of laws passed by Congress. The question is, of course, whether the laws are consistent with the fundamental postulates contained in the Constitution.

Similarly, independence of postulates can be illustrated by reference to lawmaking. It is usually considered desirable to avoid cluttering up the statute books with superfluous laws, that is, laws which are clearly redundant. However, it is quite possible that some

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* But unfortunately a set of postulates formulated for unscrupulous purposes has obvious advantages if *inconsistent*. For then anything can be proved from the postulates. (Recall our theorem in Section 8–8: "Any inconsistent set of postulates is complete.") Political leaders often seem all too aware of this in formulating campaign platforms.

laws may be needed even when they appear to be essentially logical consequences of certain provisions of the Constitution. They may cover new circumstances, or they may be designed to implement or put into actual effect certain provisions of the Constitution.

The frequent disagreements of Supreme Court justices and other legal experts on questions of constitutionality illustrate the difficulty of determining consistency or inconsistency in nonmathematical situations. It is often equally difficult to establish independence or lack of it.

Are the simple techniques of establishing consistency and independence by means of concrete interpretations or examples ever applicable in these situations? Probably the answer is "sometimes," or "with modifications." If the postulates in question can be considered analogous to a set of abstract mathematical postulates, so that they can be judged to be "satisfied" in varied circumstances, then the standard methods should be applicable, at least informally.

A familiar illustration of everyday postulates of this nature is afforded by sets of criteria, or check lists. For example, the manufacturer of a certain automobile advertises ten features which a modern car should possess. The car being advertised, of course, has all of these features, and this fact constitutes a consistency proof. (Rival cars usually fail to satisfy all of the listed criteria.)

Other common lists of criteria include qualifications for voting, characteristic symptoms of specified diseases, criteria for an ideal husband or wife, requirements for admission to or graduation from college, and necessary qualifications for a person to fill a certain position, e.g., as president of the United States. Sometimes the listed criteria may be on such a high level that no consistency proof seems possible in practice. In such circumstances we may have to admit that there is an inconsistency in the proposed criteria, or we may continue to have faith that they are consistent, and temporarily accept a partial realization of them.

Lists of criteria can also be used to illustrate the role of independence of postulates. Certainly it would seem desirable to eliminate obvious redundancies from a check list of the type we have been discussing, but independence proofs of the usual variety are hardly likely to be attempted systematically. In case of doubt as to whether a certain criterion is really independent of other criteria, it would seem reasonable to include it explicitly.

As to completeness and categoricalness, there may be more use

for these concepts than would generally be recognized, often from a negative point of view. For example, it might help with everyday arguments to realize the possibility that a certain controversial statement is neither provable nor disprovable on the basis of a given set of assumptions — i.e., that the set of assumptions is incomplete.

From a positive point of view, an informal variety of categoricalness is often desired. Roughly, the idea is to achieve a set of criteria which are jointly sufficient to determine only one kind of "model" of the criteria. In the case of the check list of modern features for an automobile, it would ordinarily be the claim of the manufacturer that the criteria are "categorical" in that they point definitely to his car as the best buy. Similarly, sponsors of a candidate for public office often are able to prepare a list of qualifications for the office which supposedly indicates with categorical definiteness that their candidate is the best man. More genuinely worth-while examples of "categoricalness" might be found in the fields of medicine or biology.

EXERCISES

1. Investigate the basis of any recent constitutionality decision of the United States Supreme Court. If there was a dissenting opinion, investigate the opposition arguments also.

2. In a debate between *A* and *B*, *A* accepts Fundamental Principles 1, 2, 3, 4 and *B* accepts 1, 2, 3 only. What would be the significance of the following possible events (considered separately)?

I. *A* proves Principle 4 is redundant in relation to Principles 1, 2, 3.
II. *B* proves Principle 4 is independent of the others.

3. The following is a quotation from the science column of the *New York Times*, July 9, 1944, concerning criteria for the pollen which causes hay fever:

"To cause hay fever a pollen must satisfy these conditions:
(1) It must contain an excitant. Most pollen doesn't.
(2) It must be borne by the wind.
(3) It must be scattered in large amounts.
(4) It must be so small that it can be readily wafted.
(5) It must come from a widely distributed plant."

As far as you can judge from this quotation, what would your opinion be concerning the consistency, independence, and "categoricalness" of these criteria?

4. Give an example of a situation from medicine or biology (or some other

field) in which it might be very useful to know that a certain set of criteria is probably "categorical."

5. Discuss the pros and cons of possible redundancies in the constitution and by-laws of a club.

6. Discuss with regard to the usual characteristics of postulates the following statement of the qualities of a good teacher, adopted by the National Conference on Education of Teachers (as reported in the *New York Times*, July 3, 1948):

"A good teacher should

(1) Possess human qualities — love of children, sympathetic understanding for all, fairness, patience, humor, and a sense of justice.

(2) Have emotional stability and adjustment.

(3) Possess outstanding native ability together with adequate professional training (five years suggested as minimum) and should like to teach.

(4) Be intellectually alert and curious.

(5) Have pleasing personality and appearance, enthusiasm, vigor, vitality, poise, and charm.

(6) Be able to develop love for the democratic way of life and be a full participating member of the democratic society.

(7) Have the ability to think critically and objectively.

(8) Have a healthy physical and mental outlook on life, with well-rounded interests.

(9) Keep up to date with modern educational trends and philosophies, and participate in professional affairs.

(10) Understand the importance of developing world citizenship and better human relationships."

7. Find and discuss a set of criteria or assumptions in some field of special interest to you.

8–12 Summary. We now recapitulate our discussion of characteristics of sets of postulates, omitting the references to applications just covered in the preceding two sections.

Every set of postulates in order to be acceptable mathematically and logically must satisfy the requirement of *consistency*. In order to establish consistency we exhibit a concrete illustration (model), based on a specified interpretation of the undefined terms used in the given postulates, which is judged to satisfy all of the postulates simultaneously. This type of consistency proof is not considered absolute.

Independence of a set of postulates is theoretically desirable for purposes of clear-cut logical organization in the development of a

mathematical subject. However, it is not usually investigated in detail, and often known redundancies in postulates are tolerated to simplify the presentation of a branch of mathematics from the psychological or pedagogical point of view. In order to establish the independence of a set of postulates already known to be consistent, we exhibit for each postulate of the set a concrete interpretation for which the postulate in question fails and the remaining postulates are all satisfied.

A set of postulates is *complete* if it is impossible to formulate any new postulate, in terms of the undefined concepts of the system, which remains independent of the original postulates. Thus any such proposed statement is theoretically either provable or disprovable in the system.

If it is possible to define and set up an isomorphism between any two models of a given set of postulates, the set is *categorical*. Categoricalness guarantees essentially that all models of the postulates have abstractly the same properties. Hence any property which can be formulated in terms of the basic undefined terms will hold good in all models or fail in all models.

A categorical set of postulates for a certain branch of mathematics furnishes a concise and adequate list of key principles for the abstract development of the subject. But a noncategorical set of postulates has the advantage of permitting the unified logical development of properties common to a variety of mathematical theories which can differ considerably among themselves.

In the next two chapters we turn our attention to some of the most important mathematical systems depending on noncategorical and incomplete sets of postulates.

CHAPTER 9 *

GROUPS, RINGS, BOOLEAN ALGEBRAS

9–1 Introduction. In this chapter we study some systems — groups, rings, Boolean algebras — which are of importance in many different parts of mathematics. Their wide applicability is partially explainable by the lack of categoricalness of their postulates, while lack of completeness is emphasized by the possibility of special varieties of these systems, which can be defined by adding suitable postulates to the various basic lists. We shall deduce fundamental properties for each type of system and also indicate relationships of the systems to each other and to fields.

9–2 Postulates for a group: Set I. A group refers to a set of elements combinable under a *single* operation and satisfying specified postulates which can be summarized as *closure*, *associative*, *identity*, and *inverse* postulates. Here we list the postulates in a stronger form than necessary, and give a weaker form in the next section.

Group Postulates: Set I

Given a set G of elements a, b, c, . . . combinable under an operation designated as o. G is called a *group*, with respect to o, if the following postulates are satisfied:

Postulate 1. *Closure.* For every a,b in G, $a\ o\ b$ is an element in G.

Postulate 2. *Associative.* For every a,b,c in G

$$(a\ o\ b)\ o\ c = a\ o\ (b\ o\ c).$$

Postulate 3. *Identity.* There exists in G an element I such that for every a in G

$$a\ o\ I = I\ o\ a = a.$$

(I is called the *identity element* of G.)

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* This chapter may be omitted without loss of continuity (except that it is needed in the last section of Chapter 10).

POSTULATE 4. *Inverse.* For every element a in G there exists an element $\hat{a}$ in G such that

$$a \, o \, \hat{a} = \hat{a} \, o \, a = I.$$

($\hat{a}$ is called the *inverse* of a and may be read "a inverse.")

The consistency of these postulates is readily shown. For example, interpret G as the set of all integers (positive, negative, zero) and o as the operation of addition. Then each postulate is satisfied, assuming our usual knowledge about the concrete theory of integers. This is an example of an *infinite group.*

For still simpler consistency examples we could use examples of a *finite group*, such as the modulo 2 system or modulo 3 system, under addition, or even the system consisting of the single integer 0, under either addition or multiplication. The latter example is trivial but it nevertheless satisfies all the postulates. We shall not investigate the question of independence of the postulates.*

We note that a group is a more general type of system than a field in two different ways: there is only one operation in a group, and the commutative postulate for that operation is not included in the group postulates. In all groups involving sets of numbers or number symbols (as in modulo systems) we are sure, however, that the commutative postulate holds. If there is any reason to suppose that the commutative postulate is redundant, we shall soon see that this is not the case. The sets of elements forming groups may be of many different types and, as we have just seen, the number of elements in a group may be finite or infinite.

Before going further, we prove two theorems about the uniqueness of identity and inverse elements. (We recall that these properties were incorporated in the postulates for fields, making the postulates stronger than necessary.)

THEOREM 1. *The identity element of a group is unique.*

Proof. Suppose there are two identity elements, I_1 and I_2. Then by Postulate 3,

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* If we were concerned with the question of independence, certain modifications in the statement of the postulates would be required. For example, in the case of an interpretation for which the identity postulate fails, Postulate 4 would need the qualification "if I exists in G." Then Postulate 4 would be satisfied automatically by the falsity of the "if" clause. In such a case a postulate is said to be satisfied *vacuously.*

$$I_1 \, o \, I_2 = I_2 \, o \, I_1 = I_1 \quad \text{(using } I_1 \text{ as } a \text{ and } I_2 \text{ as } I\text{)},$$

and also

$$I_2 \, o \, I_1 = I_1 \, o \, I_2 = I_2 \quad \text{(using } I_2 \text{ as } a \text{ and } I_1 \text{ as } I\text{)}.$$

Hence $$I_1 = I_2,$$

and the identity element must be unique.

THEOREM 2. *For every element a in a group the inverse $\hat{a}$ is unique.*

Proof. Suppose $\hat{a}_1$, $\hat{a}_2$ are both inverses of a. Then by the inverse postulate:

$$a \, o \, \hat{a}_1 = I$$

and $$a \, o \, \hat{a}_2 = I.$$

Hence $$a \, o \, \hat{a}_1 = a \, o \, \hat{a}_2.$$

Now operate on each side, on the left, by $\hat{a}_1$:

$$\hat{a}_1 \, o \, (a \, o \, \hat{a}_1) = \hat{a}_1 \, o \, (a \, o \, \hat{a}_2).$$

Using the associative postulate,

$$(\hat{a}_1 \, o \, a) \, o \, \hat{a}_1 = (\hat{a}_1 \, o \, a) \, o \, \hat{a}_2,$$

which gives

$$I \, o \, \hat{a}_1 = I \, o \, \hat{a}_2,$$

and finally

$$\hat{a}_1 = \hat{a}_2.$$

COROLLARY. *The inverse of $\hat{a}$ is a.*

EXERCISES

1. Do the following sets of elements with operations form groups? Why or why not?

(a) The set of positive and negative integers under multiplication.

(b) The set of all integers (positive, negative, and zero) under addition; also under subtraction.

(c) A modulo m addition table (where m is any positive integer).

(d) The symbols $1, 2, \ldots, m - 1$ under multiplication, modulo m, where m is a composite integer.

(e) The symbols $1, 2, \ldots, p - 1$ under multiplication, modulo p, where p is a prime integer.

(f) The set of nonzero rational numbers under multiplication; also under division.

(g) The complex fourth roots of unity, ± 1, $\pm i$ under multiplication.

(h) The positive real numbers under multiplication.

2. Prove that left and right cancellation rules hold in any group:

(a) If $a \, o \, b = a \, o \, c$, then $b = c$.

(b) If $a \, o \, c = b \, o \, c$, then $a = b$.

3. In a group containing an even number of elements, prove that there is at least one element besides the identity which is its own inverse. (*Hint:* Use the indirect method.)

9–3 Weaker postulates for a group: Set II. We now show that Postulates 3 and 4, the identity and inverse postulates, of Set I can be replaced by weaker postulates which can be called right identity and right inverse postulates. If this replacement is made, we shall call the new set of postulates Set II. Set II will then consist of Postulates 1, 2, 3′, 4′, where the new postulates are as follows:

POSTULATE 3′. *Right identity.* There exists in G an element I such that for every a in G

$$a \, o \, I = a.$$

POSTULATE 4′. *Right inverse.* For every element a in G there exists an element $\hat{a}$ in G such that

$$a \, o \, \hat{a} = I.$$

It is apparent that Postulates 3′ and 4′ are deducible respectively from Postulates 3 and 4. Hence Set II is deducible from Set I.

To show that Set I is deducible from Set II, we assume Set II and prove the following theorems.

THEOREM II–1. Right cancellation. *If* $a \, o \, c = b \, o \, c$ *in* G, *then* $a = b$.

Proof. Using a right inverse $\hat{c}$ of c which exists in G, we have

$$(a \, o \, c) \, o \, \hat{c} = (b \, o \, c) \, o \, \hat{c}.$$

$$\therefore \quad a \, o \, (c \, o \, \hat{c}) = b \, o \, (c \, o \, \hat{c}) \qquad \text{by associative postulate,}$$

$$a \, o \, I = b \, o \, I \qquad \text{by right inverse postulate,}$$

and

$$a = b \qquad \text{by right identity postulate.}$$

THEOREM II–2. *A right identity element in* G *is also a left identity element.*

Proof. We wish to show that an element I satisfying $a \, o \, I = a$ for every a in G also satisfies $I \, o \, a = a$.

To do this we note that $\hat{a}$ is available in G as a right inverse of a, and

$$(I \, o \, a) \, o \, \hat{a} = I \, o \, (a \, o \, \hat{a}) \qquad \text{by associative postulate,}$$

$$= I \, o \, I \qquad \text{by right inverse postulate,}$$

$$= I \qquad \text{by right identity postulate,}$$

$$= a \, o \, \hat{a} \qquad \text{by right inverse postulate.}$$

Hence $I \, o \, a = a$ by right cancellation (Theorem II–1).

THEOREM II–3. *A right inverse of an element a in G is also a left inverse.*

Proof. We wish to show that an element $\hat{a}$ satisfying $a \, o \, \hat{a} = I$ also satisfies $\hat{a} \, o \, a = I$. We have

$$\begin{aligned}
(\hat{a} \, o \, a) \, o \, \hat{a} &= \hat{a} \, o \, (a \, o \, \hat{a}) && \text{by associative postulate,} \\
&= \hat{a} \, o \, I && \text{by right inverse postulate,} \\
&= \hat{a} && \text{by right identity postulate,} \\
&= I \, o \, \hat{a} && \text{by Theorem II–2.}
\end{aligned}$$

Hence $\hat{a} \, o \, a = I$ by right cancellation (Theorem II–1).

The last two theorems above show that Postulates 3 and 4 are deducible from Set II and hence that Set I as a whole is deducible from Set II.

Thus Sets I and II are interdeducible, and hence logically equivalent. In checking to see whether or not a certain concrete interpretation is a group, there are often advantages in using the weaker Set II. Of course, if we are building up theorems from Set II, we immediately have available the theorems already obtained from Set I.

EXERCISES

1. Prove that the equations $x \, o \, b = a$ and $b \, o \, y = a$ have unique solutions in a group G.
2. Prove that in any group, if $a \, o \, b = c$, then $\hat{c} = \hat{b} \, o \, \hat{a}$.

9–4 Independence of the commutative postulate. Abelian groups. Permutation groups. In this section we shall justify our previous claim that the commutative postulate is independent of the standard group postulates (Set I or Set II). First we list the postulate below, and give a related definition.

POSTULATE 5. *Commutative.* For every a, b in G,

$$a \, o \, b = b \, o \, a.$$

DEFINITION. A group which satisfies the commutative postulate is called an *Abelian* (or *commutative*) group.

All of the examples of groups we have had so far have been Abelian groups. Any one of these can be used as a consistency proof for

the combined Postulates 1 to 5. For an independence proof we must also, of course, supply a concrete illustration of a group which is *not* Abelian.

For this purpose we introduce the concept of a *permutation* as a *process* of replacing each one of a finite set of symbols (e.g., letters or digits) by a symbol (not necessarily a *different* symbol) chosen from the same set.*

Suppose we start with the symbols

$$1, 2, 3,$$

and wish to replace these respectively by

$$3, 1, 2.$$

We indicate this permutation by the following notation:

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

The same permutation could be indicated by

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix},$$

since, in each case, 1 is to be replaced by 3, 2 by 1, and 3 by 2.

We now introduce the operation known as permutation "multiplication." To multiply permutations we simply find the single permutation which results by performing two permutations, one after the other. Thus, to multiply

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},$$

we see that the first permutation followed by the second sends

1 into 3, and 3 into 2, i.e., 1 into 2;
2 into 1, and 1 into 1, i.e., 2 into 1;
3 into 2, and 2 into 3, i.e., 3 into 3.

Hence the product of the two permutations can be indicated as follows:

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* This differs from the common use of "permutation" as a set of things arranged in a definite order. The latter concept we shall refer to as an *arrangement.*

$$\begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix}=\begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix}.$$

It is clear now that the product of any two permutations involving the symbols 1, 2, 3 (and no other symbols) will be another permutation on these same symbols.

The *number* of possible permutations on these symbols can easily be found by agreeing to write the first line of symbols in the *order* 1, 2, 3. Then we count the number of ways in which the corresponding places in the second line can be filled:

$$\begin{pmatrix}1 & 2 & 3\\ \text{—} & \text{—} & \text{—}\end{pmatrix}.$$

Clearly the number of ways of filling the blanks with the available symbols is the number of *arrangements* of three things taken three at a time, which is 3! or 6. In the same way, the total number of permutations on n symbols is $n!$.

We now prove that the six permutations on three symbols form a group, and furthermore show that this group is *non-Abelian*, i.e., it fails to satisfy the commutative postulate. Thus the specified group will establish the independence of the commutative postulate in relation to the standard group postulates.

THEOREM 1. *The set S_3 of all permutations on three symbols forms a group under permutation multiplication.*

Proof. The closure postulate holds in S_3 by our remark above that the product of any two permutations on 1, 2, 3 is another permutation on the same symbols.

To check the associative postulate we use the symbols $i, j, k; l, m, n; p, q, r$ in each case to designate 1, 2, 3 in some order. Hence we can write:

$$\left[\begin{pmatrix}1 & 2 & 3\\ i & j & k\end{pmatrix}\begin{pmatrix}i & j & k\\ l & m & n\end{pmatrix}\right]\begin{pmatrix}l & m & n\\ p & q & r\end{pmatrix}=\begin{pmatrix}1 & 2 & 3\\ l & m & n\end{pmatrix}\begin{pmatrix}l & m & n\\ p & q & r\end{pmatrix}=\begin{pmatrix}1 & 2 & 3\\ p & q & r\end{pmatrix}.$$

Also

$$\begin{pmatrix}1 & 2 & 3\\ i & j & k\end{pmatrix}\left[\begin{pmatrix}i & j & k\\ l & m & n\end{pmatrix}\begin{pmatrix}l & m & n\\ p & q & r\end{pmatrix}\right]=\begin{pmatrix}1 & 2 & 3\\ i & j & k\end{pmatrix}\begin{pmatrix}i & j & k\\ p & q & r\end{pmatrix}=\begin{pmatrix}1 & 2 & 3\\ p & q & r\end{pmatrix}.$$

This proves that the associative postulate is satisfied for all permutations of S_3.

The right identity postulate holds, since

$$\begin{pmatrix}1 & 2 & 3\\ i & j & k\end{pmatrix}\begin{pmatrix}i & j & k\\ i & j & k\end{pmatrix}=\begin{pmatrix}1 & 2 & 3\\ i & j & k\end{pmatrix}.$$

Thus the permutation

$$\begin{pmatrix}i & j & k\\ i & j & k\end{pmatrix}, \quad \text{that is,} \quad \begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix},$$

serves as a right identity element in S_3.

Finally, to show that the right inverse postulate holds, we merely note that

$$\begin{pmatrix}1 & 2 & 3\\ i & j & k\end{pmatrix}\begin{pmatrix}i & j & k\\ 1 & 2 & 3\end{pmatrix}=\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix}.$$

Hence the permutation $\begin{pmatrix}1 & 2 & 3\\ i & j & k\end{pmatrix}$ has a right inverse in S_3, namely,

$$\begin{pmatrix}i & j & k\\ 1 & 2 & 3\end{pmatrix}.$$

Thus all postulates for a group (Set II) are satisfied.

The above theorem can readily be generalized to show that the set of all $n!$ permutations on n symbols forms a group. Such a group is known as the *symmetric permutation group* on the given symbols. Other permutation groups are possible which contain some but not all of the $n!$ permutations of the symmetric group.

We now return to our group S_3, the symmetric group on three symbols. The six permutations of this group may be listed with the following abbreviated names as follows:

$$I=\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix}, \quad a=\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix}, \quad b=\begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix},$$

$$c=\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}, \quad d=\begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}, \quad e=\begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix}.$$

The next theorem shows that the commutative postulate is *not* satisfied by the group S_3 and thus establishes our claim that the commutative postulate is independent of the standard group postulates.

THEOREM 2. *The symmetric group S_3 on three symbols is non-Abelian.*

Proof. It is sufficient to exhibit a pair of permutations of S_3 which produce different "products" depending on the order of the factors. Consider the permutations a and d:

$$ad = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = e.$$

$$da = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = b.$$

Thus $ad \neq da$, and the group S_3 must be non-Abelian.

Theorem 2 not only proves that the commutative postulate is *independent* of the group postulates, but it also shows that the group postulates are *incomplete*.

A more obvious proof of incompleteness of the group postulates can be given by formulating other postulates which are clearly independent of Set I or Set II, such as "there are exactly *two* elements in the set G," or "there is a *finite number* of elements in G." The latter postulate taken in conjunction with the standard postulates leads to the theory of *finite groups*. Permutation groups have an important role in the development of this theory.*

The theory of finite groups in turn has useful applications in the theory of algebraic equations, including significant bearings on ruler-and-compass constructibility criteria. These methods go back to one of the originators of the group concept, the French mathematical genius, Galois (1811–1832).†

Exercises

1. Construct a complete multiplication table for the permutations I, a, b, c, d, e of the symmetric group S_3. (Enter $ab = c$, $ad = e$, etc.)

2. Prove that the permutations I and a themselves form a group under permutation multiplication. Since I and a belong to the symmetric permutation group S_3, they are said to form a *subgroup* of the latter.

3. Find other two-element subgroups of S_3.

4. Construct a multiplication table for I, c, and d, and show that these permutations form a subgroup of S_3. Call this subgroup H_3. Is H_3 *Abelian* or *non-Abelian?*

5. (a) Construct a multiplication table for the three complex cube roots of unity, 1, ω, ω^2, and show that they form a group.

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* A famous theorem due to the English mathematician Cayley (1821–1895) states that every finite group is isomorphic to a certain permutation group.

† See E. T. Bell, *The Development of Mathematics*, McGraw-Hill, 1940, Chapter 10.

$$\left(\text{Here } \omega = \cos 120° + i \sin 120° = -\frac{1}{2} + \frac{i}{2}\sqrt{3},\right.$$
$$\left.\text{and } \quad \omega^2 = \cos 240° + i \sin 240° = -\frac{1}{2} - \frac{i}{2}\sqrt{3}.\right)$$

(b) Show that the group C of cube roots of unity is isomorphic to the group H_3 of Exercise 4. (That is, show that a pairing of elements of the two groups can be set up such that the multiplication table for one group becomes a "translation" of the table for the other group.)

6. Show that the group H_3 of Exercise 4 is isomorphic to the modulo 3 addition group.

7. Prove that the modulo 4 addition group is isomorphic to the multiplication group of the fourth roots of unity $(1, i, -1, -i)$.

8. Find a subgroup of the modulo 4 addition group, and a multiplication group isomorphic to this.

9. Prove that the set of postulates 1,2,3′,4′,5 for an Abelian group is logically equivalent to the following set of four postulates:

(i) The closure postulate.

(ii) For every a,b,c in G: $(a \circ b) \circ c = (c \circ b) \circ a$.

(iii) The right identity postulate.

(iv) The right inverse postulate.

9–5 Fields described in group terminology. We devote this section to the following theorem characterizing fields in terms of Abelian groups.

THEOREM. *A field is logically equivalent to a set (K) of at least two distinct elements, combinable under two operations (designated as $+$ and $\times$) such that the following conditions are satisfied:*

(1) *The elements of K form an Abelian group (G_+) under the operation $+$ (with the identity element of G_+ designated as a "zero" element).*

(2) *The nonzero elements of K form an Abelian group $(G_\times)$ under the operation $\times$ (with the identity element of $G_\times$ designated as a "unity" element).*

(3) *The left and right distributive postulates hold for every a, b, c in K,*

$$a(b + c) = ab + ac,$$
$$(a + b)c = ac + bc.$$

Proof. I. The above description of a field is clearly deducible from the field postulates given in Chapter 6 as the "postulates of rational operations" (Set A).

II. Now we assume that we have a set of elements satisfying the

above description, and show that it must satisfy the postulates of Set A.

Since the elements of K form an Abelian group, under $+$, we know that Postulates 1 to 5 of Set A hold (the required uniqueness of 0, and opposites, following by theorems we have proved for any group).

Likewise, we know that the elements of K, *excluding* 0, satisfy Postulates 1′ to 5′ of Set A. Postulate 6 (the left distributive postulate) is satisfied since it duplicates one half of condition (3) above.

What remains to be shown is that the zero element can be included as an element satisfying Postulates 1′ to 4′. 0 is not required to have an inverse element under multiplication in Postulate 5′, so that this postulate is not involved here.

From the left distributive postulate, assumed for all a, b, c in K, we can prove the left distributive rule for subtraction just as in the development of theorems in Chapter 6. (This involves only the additional use of Postulates 1 to 5, which are already available.) Hence, for any element $a \neq 0$ — and there must be such an element in K by our assumption of at least two distinct elements — we have

$$a \cdot 0 = a(b - b) = ab - ab = 0.$$

Thus $a \cdot 0$ is in K.

In the same way, from the right distributive postulate, we can obtain for $a \neq 0$,

$$0 \cdot a = 0.$$

Also we can write (still using $a \neq 0$)

$$0 \cdot 0 = 0(a - a) = 0a - 0a = 0.$$

This shows that the closure postulate holds for multiplication when one or both of the elements a, b is 0. It also shows that the commutative and associative postulates hold when one or more of the elements is 0, and that $0 \cdot 1 = 0$.

We still must check that the unity element and zero element are distinct from each other. This is easily done indirectly. For suppose we had

$$1 = 0;$$

there would then be in K an element $a \neq 0$, making

$$a \cdot 1 = a \cdot 0,$$
$$a = 0,$$

which is a contradiction. This completes the proof.

9–6 Rings. We now consider a new type of system with two operations, known as a *ring*, which can be thought of as a generalization of a field. The generalization is accomplished by relaxing some of the restrictions imposed by the field postulates. (This is like the process we have already suggested for weakening a single postulate by removing such conditions as uniqueness or commutativity.) Here the restrictions to be removed are those postulates of a field calling for (1) commutative multiplication, (2) existence of a unity element, and (3) existence of inverse elements with respect to multiplication (i.e., reciprocals).

The result can be summarized as follows.

POSTULATES FOR A RING

Given a set R of elements $a, b, c, \ldots$, combinable under two operations designated as $+, \times$. R is called a *ring* with respect to $+, \times$ if the following postulates are satisfied:

POSTULATE 1. *Closure for addition.* For every a, b in R,

$$a + b \text{ is an element of } R.$$

POSTULATE 2. *Associative addition.* For every a, b, c in R,

$$(a + b) + c = a + (b + c).$$

POSTULATE 3. *Right identity for addition.* There exists in R an element 0 such that for every a in R

$$a + 0 = a.$$

POSTULATE 4. *Right inverse for addition.* For every a in R there exists an element $\overline{a}$ in R such that

$$a + \overline{a} = 0.$$

POSTULATE 5. *Commutative addition.* For every a, b in R,

$$a + b = b + a.$$

(Postulates 1 to 5 require that R be an Abelian group under addition.)

POSTULATE 6. *Closure for multiplication.* For every a, b in R,

$$ab \text{ is an element of } R.$$

POSTULATE 7. *Associative multiplication.* For every a, b, c in R

$$(ab)c = a(bc).$$

POSTULATE 8. *Left distributive.* For every a, b, c in R,

$$a(b + c) = ab + ac.$$

POSTULATE 9. *Right distributive.* For every a, b, c in R,

$$(a + b)c = ac + bc.$$

Of course, every field satisfies these postulates, so *every field is a ring*, and we need no new proof of consistency for the ring postulates. As obvious illustrations of rings which are *not* fields, we mention the following.

Illustration 1. The (infinite) set of *all* even integers (positive, negative, 0) is a ring under the usual operations. Note that there is no unity element in the ring.

Illustration 2. The (finite) modulo 4 system under addition and multiplication is a ring. The operation tables are as follows:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

×	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

This is a ring which contains a unity element, but not every nonzero element of the system has an inverse (2 does not have an inverse). Also since $2 \cdot 2 = 0$, the ring has divisors of zero.

Rings which satisfy certain additional postulates besides 1 to 9 above are often given special names, as we indicate below.

POSTULATES FOR SPECIAL TYPES OF RINGS

POSTULATE 10. *Commutative multiplication.* For every a, b in R,

$$ab = ba.$$

(A ring which satisfies this postulate is called a *commutative ring.*)

POSTULATE 11. *Two-sided identity for multiplication.* There exists in R a unity element designated as 1, such that for every a in R,

$$1a = a1 = a.$$

(A ring which satisfies this postulate is called a *ring with unity element.*)

POSTULATE 12. *Idempotent.* For every a in R, a has the idempotent property

$$a \cdot a = a.$$

(A ring which satisfies this postulate is called a *Boolean ring.**)

POSTULATE 13. There exist in R nonzero elements a, b such that

$$ab = 0.$$

(A ring which satisfies this postulate is called a *ring with divisors of zero.*)

A field can now be characterized readily as a special type of ring in the following way.

THEOREM F. *A field is logically equivalent to a commutative ring of at least two distinct elements, including a unity element, in which reciprocals of nonzero elements exist.*

Of course any known field gives a consistency proof for Postulates 1 to 11 inclusive. Postulates 1 to 12 are consistent, for they are all satisfied by the example of an idempotent field (used in Chapter 8), i.e., the modulo 2 system. Postulate 13 is consistent with Postulates 1 to 11, as shown by the ring in Illustration 2, which we know has divisors of zero. We shall see in the next section that Postulate 13 is also consistent with the first twelve postulates.

The independence of these extra postulates in relation to the general ring postulates now follows if we produce examples showing their nonredundancy. Illustration 1, which contains no unity element, is such an example for Postulate 11, while this illustration, or Illustration 2, or almost any field serves the purpose for Postulate 12. Also we know that any field fails to satisfy Postulate 13.

It is possible also to give a nonredundancy example for the commutative multiplication postulate, but we shall not attempt that here. The standard illustration is to use systems of *matrices.*

We note from Theorem F just above that certain field theorems will hold for rings also. Such theorems will be precisely those which do not involve the properties used in Theorem F to characterize a field as a special type of ring.

In particular, we obtain as general ring theorems those field theorems which can be proved without use of any of the following

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* First defined by the American mathematician M. H. Stone in 1936.

field postulates: (1) commutative multiplication, (2) existence of a unity element, (3) existence of reciprocals. Thus we obtain immediately the following familiar theorems concerning opposites and zero, among others.

Theorems for General Rings

Theorem 1. *For every a and b in a ring R the equation $x + b = a$ has a unique solution in R, namely,* $x = a + \bar{b}$.

Definition. $a - b = a + \bar{b}$.

Theorem 2. Left and right distributive rules for multiplication with respect to subtraction. *For every* a, b, c in R,

$$a(b - c) = ab - ac,$$

and

$$(a - b)c = ac - bc.$$

Theorem 3. $a \cdot 0 = 0 \cdot a = 0$.

Theorem 4. $a(\bar{b}) = \overline{ab}$.

Theorem 5. $\bar{a}\bar{b} = ab$.

Exercises

1. Prove that the set of all integers under addition and multiplication forms a ring, and describe its properties.

2. Prove that a modulo m system, where m is a composite integer, is a ring with unity element and divisors of zero. Illustrate by use of the modulo 6 system.

3. Prove that a Boolean ring with two elements is isomorphic to the field F_2, that is, the modulo 2 system.

4. Do the following systems form rings under the usual operations? If so, describe their properties. (a) The set of all rational numbers. (b) The set of real numbers of the form $m + n\sqrt{2}$, where m and n are allowed to vary over the set of all integers. (c) The set of three complex cube roots of unity, 1, ω, ω^2 (see Exercise 5 of Section 9–4).

5. Can it be proved that a multiplication cancellation law holds for all rings? In other words, is it a general ring theorem that $ab = ac$ implies $b = c$ when $a \neq 0$?

6. Prove the right distributive part of Theorem 2 above. (Special attention is needed since this was not listed as a separate theorem in the Chapter 6 development.)

9–7 Boolean rings. As indicated by Postulate 12 of the preceding section, a Boolean ring is a ring in which every element is idempotent.

Boolean rings have other interesting properties and will furnish a convenient transition to Boolean algebras. We proceed to deduce some Boolean ring theorems which we label "BR" for future reference.

THEOREMS FOR BOOLEAN RINGS

THEOREM BR-1. *In a Boolean ring B, every element is its own additive inverse; i.e., for every a in B,*

$$a + a = 0.$$

Proof. For every a in B,

$$\begin{aligned} a + a &= (a + a)(a + a) && \text{by idempotent postulate,} \\ &= a(a + a) + a(a + a) && \text{by right distributive postulate,} \\ &= (aa + aa) + (aa + aa) && \text{by left distributive postulate,} \\ &= (a + a) + (a + a) && \text{by idempotent postulate,} \end{aligned}$$

i.e., $0 + (a + a) = (a + a) + (a + a)$.

$\therefore \quad a + a = 0$ by right cancellation in the additive group of B.

COROLLARY 1. If $a + b = 0$, then $a = b$.

COROLLARY 2. $a + b = a - b$.

COROLLARY 3. If $a + b = c$, then $a = c + b$.

THEOREM BR-2. *A Boolean ring is a commutative ring.*

Proof. It is sufficient by Corollary 1 above to show that for every a, b in B we must have $ab + ba = 0$.

$$\begin{aligned} \text{Now} \quad a + b &= (a + b)(a + b) && \text{by idempotent postulate,} \\ &= a(a + b) + b(a + b) && \text{by right distributive postulate,} \\ &= (aa + ab) + (ba + bb) && \text{by left distributive postulate,} \\ &= (a + ab) + (ba + b) && \text{by idempotent postulate} \\ &= (ab + ba) + (a + b) && \text{by commutative and associative addition postulates.} \end{aligned}$$

$\therefore \quad ab + ba = 0$ as in similar step of Theorem BR-1.

$\therefore \quad ab = ba.$

THEOREM BR-3. *A Boolean ring of two elements is logically equivalent to an idempotent field.*

Proof. We have seen already that any idempotent field F_I must contain exactly two elements (the zero and unity elements). Hence F_I is a two-element ring satisfying the idempotent postulate, that is, a Boolean ring of two elements.

On the other hand, given a Boolean ring B of two elements a and b, we must have one of these, say a, act as a zero element, and thus

$$a \cdot a = a \cdot b = b \cdot a = a.$$

Also by the idempotent postulate,

$$b \cdot b = b.$$

Thus b acts as a unity element. Also multiplication is commutative and the required reciprocal exists for the nonzero element b (i.e., b itself). Hence all of the postulates for an idempotent field are satisfied.

THEOREM BR-4. *A Boolean ring B containing more than two elements is a ring with divisors of zero.*

Proof. By hypothesis, B contains distinct elements a, b such that $a \neq 0$ and $b \neq 0$. Then also $a + b$ is an element of B by closure, and

$$a + b \neq 0.$$

(If $a + b$ were 0, then by Corollary 1 of Theorem BR-1 we would have $a = b$.) But ab is also an element of B, and

$$\begin{aligned} ab(a+b) &= (ab)a + (ab)b && \text{by left distributive postulate,} \\ &= (aa)b + a(bb) && \text{by Theorem BR-2 and associative multiplication postulate,} \\ &= ab + ab && \text{by idempotent postulate.} \\ \therefore\quad ab(a+b) &= 0 && \text{by Theorem BR-1.} \end{aligned}$$

This shows that if $ab \neq 0$, then ab and $a + b$ are divisors of zero in B. On the other hand, if $ab = 0$, then both a and b are themselves divisors of zero.

In connection with Theorem BR-3 above it may be of interest to give an interpretation from logic for a Boolean ring of two elements.

For this purpose think of the system of all propositions, i.e., statements which can be classified as "true" or "false." If p and q are propositions, interpret pq as the *conjunction* of p, q and let $p + q$ be the *complete disjunction* of p, q corresponding to the use of the word "or" in an *exclusive* sense. Then:

pq is true if and only if p and q are both true.
$p + q$ is true if and only if *one and only one* of the propositions p, q is true.

Thus we can make the following operation tables (which are just

another form of truth value tables, but written like algebraic operation tables).

+	false	true
false	false	true
true	true	false

×	false	true
false	false	false
true	false	true

It is clear that this system of true and false propositions is isomorphic to the modulo 2 system under the following pairing:

false 0,
true . . . . . 1.

If we "translate" false and true as suggested by this pairing the above tables become the familiar modulo 2 tables:

+	0	1
0	0	1
1	1	0

×	0	1
0	0	0
1	0	1

Accordingly, it must be that our given system of propositions* is an example of an idempotent field, i.e., of a Boolean ring of two elements.

We must not jump to the conclusion that *all* Boolean rings are equivalent to idempotent fields, and restricted to two elements. Although the postulates for an idempotent field are categorical, this is not the case for the Boolean ring postulates. In fact, it is possible to have either a finite or infinite number of elements in a Boolean ring. If finite, the number of elements can be shown to be equal to a power of 2. These statements will be better understood after investigating the related topic of Boolean algebras.

As a possible topic for further postulational study, we exhibit below (in abbreviated form) a weak set of postulates for Boolean rings, i.e., a set equivalent to the set obtained by adjoining the idempotent postulate to the general ring postulates. The postulates in this set have been shown to be independent.†

Weak Postulates for Boolean Rings

1. Closure for addition.
2. Associative addition.

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* Strictly speaking, it is a system of *categories* of propositions "true" and "false." The individual propositions themselves are irrelevant.

† E. R. Stabler, "Sets of Postulates for Boolean Rings," *American Mathematical Monthly*, Vol. 48, 1941, pp. 20–28.

3. Left cancellation for addition.
4. Right cancellation for addition.
5. Closure for multiplication.
6. Associative multiplication.
7. Left distributive postulate.
8. Right distributive postulate.
9. Idempotent postulate.

9–8 Boolean algebra postulates. Abstract systems, known as *Boolean algebras*, are closely related to the (concrete) logic of propositions and classes discussed in Chapters 3 and 4. The name "Boolean" is in honor of George Boole (1815–1864), British logician and mathematician, who is generally considered the founder of modern symbolic logic. Boole was Whitehead and Russell's most famous predecessor.*

The postulates for Boolean algebras which we list here are based on a well-known set given by the American mathematician and postulational specialist, E. V. Huntington.† However, the associative postulates are not found in Huntington's list. They are redundant, but we include them to simplify the presentation, as their proofs are rather tedious.

The postulates show complete duality between properties with respect to two fundamental operations. In particular, we should note the presence of the novel *addition* distributive postulate,

$$a \vee bc = (a \vee b)(a \vee c),$$

as well as the usual *multiplication* distributive postulate,

$$a(b \vee c) = ab \vee ac.$$

We list the postulates in the familiar parallel column form.

Postulates for a Boolean Algebra

Undefined concepts: Set H of elements $a, b, c, \ldots$ combinable under two operations designated as $\vee$ (Boolean addition) and $\times$ (multipli-

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* Boole's outstanding work in logic is *The Laws of Thought*. See the Bibliography.

† E. V. Huntington, "Sets of Independent Postulates for the Algebra of Logic," *Transactions of the American Mathematical Society*, Vol. 5, 1904, pp. 288–309. Since the appearance of this article many equivalent postulate sets have been formulated.

cation). It is understood that for each postulate $a, b, c, \ldots$ are given as elements of H.

POSTULATE 1.
For every a, b in H, $a \vee b$ is in H.

POSTULATE 1′.
For every a, b in H, ab is in H.

POSTULATE 2.
$a \vee b = b \vee a$.

POSTULATE 2′.
$ab = ba$.

POSTULATE 3.
$(a \vee b) \vee c = a \vee (b \vee c)$.

POSTULATE 3′.
$(ab)c = a(bc)$.

POSTULATE 4.
$a \vee bc = (a \vee b)(a \vee c)$.

POSTULATE 4′.
$a(b \vee c) = ab \vee ac$.

POSTULATE 5.
There exists in H an element Z such that for every a in H, $a \vee Z = a$.
(Z is called the *zero* element.)

POSTULATE 5′.
There exists in H an element U such that for every a in H, $aU = a$.
(U is called the *universal* element or *universe*.)

POSTULATES 6,6′. For every element a of H, there exists an element a' in H such that

$$a \vee a' = U \quad \text{and} \quad aa' = Z.$$

(a' is called the *complement* of a.)

As a consistency proof for the above postulates let us interpret H as a set consisting of *all possible subclasses of a certain fixed collection of objects* (including the whole collection as a subclass of itself). The collection might be finite (such as the collection of all chairs in a certain room) or infinite (such as the collection of all integers). It is important to realize that each "element" in the set H becomes interpreted now as a *class*. The operation $\vee$ will be interpreted as *logical addition* (or *union*) of classes, and the operation $\times$ as *logical multiplication* (or *intersection*) of classes.*

With this interpretation, Postulates 1 and 1′ are satisfied, since the union and intersection of any two classes in H must again be a

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* For reference we repeat here two definitions of Chapter 4. The *logical sum* or *union* of two classes is the class whose members belong to at least one of the given classes. The *logical product* or *intersection* of two classes is the class whose members belong to *both* of the given classes.

class of H. Postulates 2,2′, 3,3′, and 4,4′ are the commutative, associative, and distributive properties already listed as properties of logical addition and logical multiplication of classes at the end of Section 4–5.

The *zero* element in Postulate 4 is the "empty" class, consisting of none of the objects in the given collection. (This is arbitrarily thought of, nevertheless, as one of the given subclasses.) The universal element in H is taken as the subclass consisting of the whole given collection of objects. For each subclass A (in H) we interpret the required complement as the class of objects (in H) not belonging to A. Then Postulates 5,5′, 6,6′ are satisfied (as indicated by corresponding properties in the list in Section 4–5).

It is a familiar principle of the theory of combinations that the total number of classes (combinations) which can be formed from n objects taken $0,1,2,\ldots,n$ at a time, is 2^n. Thus if we use n objects in the given collection, the total number of classes which comprise the set H will be 2^n. On the other hand, if the given collection of objects is infinite, then, of course, the number of possible subclasses of the collection is infinite. This shows the great variety possible in the models of a Boolean algebra, when we interpret set H as a set of classes. Any model of this type can be referred to as an *algebra of classes.*

We can also suggest a model of the postulates which could be called an *algebra of propositions.* This is like the system of propositions, classified in the categories of "true" and "false," used in the preceding section as an example of a Boolean ring — with the difference that the operation $\vee$ (replacing $+$) is interpreted as ordinary disjunction (in accordance with our previous notation for propositions).

Under this interpretation we see that the closure postulates 1 and 1′ are satisfied. This is because the truth value of propositions $a \vee b$ and ab is determined by the truth values of propositions a and b separately. Postulates 2,2′, 3,3′, 4,4′ are satisfied because of the corresponding equivalences of propositions stated in Section 3–10. Postulate 5 holds if we assign Z to mean the category "false." This has the required property, since the disjunction of proposition a with a false proposition must agree in truth value with a itself. Likewise Postulate 5′ is satisfied if we assign U to mean the category "true." Next we take a' to mean the negation of proposition a, i.e., $\sim a$. Then Postulates 6 and 6′ are satisfied in accordance with the law of excluded middle and the law of contradiction, respectively.

Thus our *algebra of propositions* becomes a model of a two-element Boolean algebra. The operation tables are simply the following:

$\vee$	false	true
false	false	true
true	true	true

$\times$	false	true
false	false	false
true	false	true

It is apparent then that the algebra of propositions is a very restricted model of a Boolean algebra in comparison with *algebras of classes.*

The close similarity between the algebra of propositions and the system of propositions used as a model of a Boolean ring (the multiplication tables are the same in each case) suggests that there should be a close relation between Boolean rings and Boolean algebras. This is in fact the case. We proceed to show that a Boolean algebra can be considered logically equivalent to a Boolean ring with unity element.

9–9 Boolean algebra theorems. We now deduce theorems from the Boolean algebra postulates. Our aim is to prove that on defining a new operation in a Boolean algebra, to be known as ring addition and designated as $+$, every Boolean algebra becomes a Boolean ring.

In the next section we shall reverse the process by starting with the Boolean ring postulates (together with the unity element postulate) and show that we can arrive at a Boolean algebra.

The theorems in this section will be labeled "BA" to distinguish them from the Boolean ring ("BR") theorems. Most of the "BA" theorems prior to the introduction of ring addition will be in parallel column form, because of the complete duality between the operations of multiplication and Boolean addition in the Boolean algebra postulates. We need give proofs in one column only.*

THEOREMS FOR A BOOLEAN ALGEBRA

THEOREM BA-1. *The element Z is unique.*

THEOREM BA-1′. *The element U is unique.*

Proof. If there were two such elements Z_1 and Z_2, then

* Theorems 3,3′ and 5,5′ are exceptions. These make double use of Postulates 6,6′ and can be called self-dual.

$Z_1 \vee Z_2 = Z_1$ by zero element postulate,

$Z_2 \vee Z_1 = Z_2$ by zero element postulate.

But

$Z_1 \vee Z_2 = Z_2 \vee Z_1$ by commutative postulate.

$\therefore \quad Z_1 = Z_2.$

THEOREM BA-2. $Z \vee a = a$. THEOREM BA-2′. $Ua = a$.

Proof. This follows from the commutative and zero element postulates.

THEOREMS BA-3,3′. *For every a in H, the complement a′ is uniquely determined by the conditions in Postulates 6,6′.*

Proof. Suppose a_1' and a_2' both meet the requirements of Postulates 6,6′. Then we have both

$$aa_1' = Z, \quad aa_2' = Z,$$

making

$$\left.\begin{aligned} a_1' &= a_1' \vee aa_2' \\ a_2' &= a_2' \vee aa_1' \end{aligned}\right\}$$ by zero element postulate.

But

$$\begin{aligned} a_1' \vee aa_2' &= (a_1' \vee a)(a_1' \vee a_2') && \text{by distributive postulate (for } \vee \text{ with respect to } \times), \\ &= (a \vee a_1')(a_1' \vee a_2') && \text{by commutative postulate,} \\ &= U(a_1' \vee a_2') && \text{by complement postulate,} \\ &= a_1' \vee a_2' && \text{by Theorem BA-2′.} \end{aligned}$$

This makes $a_1' = a_1' \vee a_2'$.

By a similar proof, we get

$$a_2' = a_2' \vee a_1',$$

whence $a_1' = a_2'$ by commutative postulate.

THEOREM BA-4. $Z' = U$. THEOREM BA-4′. $U' = Z$.

Proof. U is an element which meets the requirements for Z' since

$Z \vee U = U$ by Theorem BA-2

and $Z \cdot U = Z$ by universal postulate.

But by the preceding theorem such an element is unique.

$\therefore \quad Z' = U.$

THEOREMS BA-5,5′. $(a')' = a$ (usually written $a'' = a$).

Proof. Since
$$a \vee a' = U$$
and
$$aa' = Z,$$
we have by commutative postulates for $\vee$ and $\times$:
$$a' \vee a = U,$$
$$a'a = Z.$$
Thus a satisfies conditions for $(a')'$ and by Theorems BA-3,3′ we must have
$$(a')' = a.$$

COROLLARY. $a' \vee a = U$.

COROLLARY. $a'a = Z$.

THEOREM BA-6. $a \vee a = a$.

THEOREM BA-6′. $aa = a$.

Proof.

$a \vee a = (a \vee a)U$ by universal postulate,

$= (a \vee a)(a \vee a')$ by complement postulate,

$= a \vee aa'$ by distributive postulate (4),

$= a \vee Z$ by complement postulate,

$= a$ by zero element postulate.

THEOREM BA-7. $a \vee U = U$.

THEOREM BA-7′. $aZ = Z$.

Proof.

$a \vee U = (a \vee U)U$ by universal postulate,

$= (a \vee U)(a \vee a')$ by complement postulate,

$= a \vee Ua'$ by distributive postulate (4),

$= a \vee a'$ by Theorem BA-2′,

$= U$ by complement postulate.

COROLLARY. $U \vee a = U$.

COROLLARY. $Za = Z$.

THEOREM BA-8. $a \vee ab = a$.

THEOREM BA-8′. $a(a \vee b) = a$.

Proof.

$a \vee ab = aU \vee ab$ by universal postulate,

$= a(U \vee b)$ by distributive postulate (4′),

$= aU$ by Corollary of Theorem BA-7,

$= a$ by universal postulate.

The next two theorems are right distributive laws. The proof of Theorem BA-9′ is familiar from the proof of the corresponding theorem for a field (with + replacing $\vee$).

THEOREM BA-9.
$ab \vee c = (a \vee c)(b \vee c)$.

THEOREM BA-9′.
$(a \vee b)c = ac \vee bc$.

Proof. This is left as an exercise.

The next two theorems are DeMorgan's laws in abstract form.

THEOREM BA-10.
$(a \vee b)' = a'b'$.

THEOREM BA-10′.
$(ab)' = a' \vee b'$.

Proof. It is sufficient by Theorems BA-3,3′ to prove that both of the following hold:

(i) $(a \vee b) \vee (a'b') = U$.
(ii) $(a \vee b)a'b' = Z$.

Proof of (i):

$(a \vee b) \vee a'b' = [(a \vee b) \vee a'][(a \vee b) \vee b']$
by addition distributive postulate,
$= [(b \vee a) \vee a'][a \vee (b \vee b')]$
by commutative and associative postulates,
$= [b \vee (a \vee a')][a \vee (b \vee b')]$
by associative postulate,
$= [b \vee U][a \vee U]$
by complement postulate,

$= U \cdot U$ by Theorem BA-7,
$= U$ by universal postulate.

Proof of (*ii*): This is left as an exercise.

Now we define the new operation of *ring addition*, to be designated by the usual symbol $+$. In forming the definition, we can get some help from our propositional interpretation. We saw that the concrete system of true and false propositions is a Boolean algebra when Boolean addition, $\vee$, is interpreted as ordinary disjunction. But the system is a Boolean ring when ring addition, $+$, is interpreted as *complete* disjunction. The complete disjunction of two propositions p, q can be expressed in terms of ordinary disjunction as

$$p(\sim q) \vee (\sim p)q.$$

This suggests for the abstract ring sum of two elements a, b the corresponding expression,

$$ab' \vee a'b.$$

DEFINITION BA-1. $a + b = ab' \vee a'b$.

We proceed to prove the theorems needed to establish our claim that every Boolean algebra is a Boolean ring with respect to the operations of multiplication and ring addition.

We drop the parallel column form of arrangement, as we are not interested here in the operation dual to $+$ (although such an operation could be defined and theorems concerning it listed in a parallel column). We include certain theorems which will be needed for the reverse deductive process in the next section.

THEOREM BA-11. *For every a, b in H, $a + b$ is an element of H.*

Proof. $a + b = ab' \vee a'b$ by Definition BA-1.
a' and b' belong to H by complement postulate.
$\therefore$ $ab' \vee a'b$ is in H by closure postulates for multiplication and Boolean addition.

THEOREM BA-12. $a + U = a'$.

Proof. $a + U = aU' \vee a'U$ by Definition BA-1,
$= aZ \vee a'U$ by Theorem BA-4,

$= Z \vee a'$ by Theorem BA-7′ and universal postulate,

$= a'$ by Theorem BA-2.

THEOREM BA-13. $a + Z = a$.

Proof. $a + Z = aZ' \vee a'Z$ by Definition BA-1,

$= aU \vee Z$ by Theorems BA-4 and BA-7′,

$= a \vee Z$ by universal postulate,

$= a$ by zero element postulate.

THEOREM BA-14. $a + a = Z$.

Proof. $a + a = aa' \vee a'a$ by Definition BA-1,

$= Z \vee Z$ by complement postulate and Theorems BA-5,5′,

$= Z$ by zero element postulate.

THEOREM BA-15. $a + b = b + a$.

Proof. This is left as an exercise.

Before proving the associative rule for ring addition we insert the following "lemma" or preliminary theorem.

LEMMA BA-1. $(a + b)' = ab \vee a'b'$.

Proof. $(a + b)' = (ab' \vee a'b)'$ by Definition BA-1,

$= (ab')' \cdot (a'b)'$ by Theorem BA-10,

$= (a' \vee b'')(a'' \vee b')$ by Theorem BA-10′,

$= (a' \vee b)(a \vee b')$ by Theorems BA-5,5′,

$= (a' \vee b)a \vee (a' \vee b)b'$ by distributive multiplication postulate,

$= (a'a \vee ba) \vee (a'b' \vee bb')$ by Theorem BA-9′,

$= (Z \vee ba) \vee (a'b' \vee Z)$ by Theorems BA-5,5′ and complement postulate,

$= ba \vee a'b'$ by zero element properties,

$= ab \vee a'b'$ by commutative multiplication postulate.

THEOREM BA-16. $(a + b) + c = a + (b + c)$.

Proof. $(a+b)+c = (a+b)c' \vee (a+b)'c$
by Definition BA-1,
$= (ab' \vee a'b)c' \vee (ab \vee a'b')c$
by Definition BA-1 and Lemma BA-1,
$= [a(b'c') \vee a'(bc')] \vee [a(bc) \vee a'(b'c)]$
by Theorem BA-9′ and associative multiplication postulate.

The remainder of the proof is to show that the right side of the desired equality gives the same result. This is left as an exercise.

THEOREM BA-17. $a(b+c) = ab + ac$.

Proof. We show that the left side and right side are equal to the same expression, as follows:

$a(b+c)=a(bc' \vee b'c)$ by Definition BA-1,
$=a(bc') \vee a(b'c)$ by distributive multiplication postulate.

$ab+ac=ab(ac)' \vee (ab)'(ac)$ by Definition BA-1,
$=ab(a' \vee c') \vee (a' \vee b')(ac)$ by Theorem BA-10′,
$=[(ab)a' \vee (ab)c'] \vee [a'(ac) \vee b'(ac)]$ by distributive rules,
$=[(a'a)b \vee (ab)c'] \vee [(a'a)c \vee b'(ac)]$ by commutative and associative postulates,
$=[Zb \vee (ab)c'] \vee [Zc \vee b'(ac)]$ by complement postulate,
$=[Z \vee (ab)c'] \vee [Z \vee b'(ac)]$ by zero element properties,
$=(ab)c' \vee b'(ac)$
$=a(bc') \vee a(b'c)$ by associative and commutative postulates.

$\therefore \quad a(b+c)=ab+ac.$

THEOREM BA-18. $(a+b)c = ac + bc$.

Proof. This follows immediately (as in the case of a field) from Theorem BA-17 and the commutative multiplication postulate.

THEOREM BA-19. *With respect to the operations of multiplication and ring addition the elements of a Boolean algebra form a Boolean ring with unity.*

Proof. Postulates 1 to 9 for a general ring (as given in Section 9–6) are satisfied respectively because of Theorems BA-11, BA-16, BA-13, BA-14, BA-15, Postulates 1′ and 3′ for Boolean algebras, Theorems BA-17 and BA-18. Finally, the idempotent postulate is satisfied because of Theorem BA-6′, and the ring has a unity element, namely, the universal element, by Postulate 5′ in the Boolean algebra list.

It is significant to note the meaning of ring addition for an interpretation of a Boolean algebra as an algebra of classes. In this case, we recall that a Boolean sum of two classes, A and B, is simply the logical sum, or union, of A and B. According to the definition for ring addition, we have

$$A + B = AB' \vee A'B.$$

This indicates that $A + B$ is the union of the classes AB' and $A'B$, and is formed by taking the total membership of these classes. But AB' is the intersection of class A with the complement of B. Thus it consists of all members of A that are not members of B. Similarly, $A'B$ consists of all members of B that are not members of A. Accordingly, the class $A + B$ is the class of all elements belonging to *one or the other* of the classes A and B, *but not both* — a class which is known as the *symmetric difference* of A and B.

In the case of two *disjoint* classes (i.e., mutually exclusive classes, classes with no common members), we see that the symmetric difference is the same as the union, that is,

$$A + B = A \vee B \quad \text{if} \quad AB = Z.$$

The symmetric difference of two classes can be obtained by deleting the common membership, or intersection, of the classes from their union. Accordingly, the union can be obtained by putting the intersection back with the symmetric difference. Thus

$$A \vee B = (A + B) \vee AB$$

and since $A + B$ and AB are disjoint classes, we can change the wedge on the right to a plus sign, obtaining

$$A \vee B = (A + B) + AB.$$

This concrete result for classes suggests the following theorem for abstract Boolean algebras.

THEOREM BA-20. $a \vee b = (a + b) + ab$.

Proof. Here it is convenient to use the Boolean ring property that each element is its own inverse with respect to ring addition. This, of course, permits transposing terms in an equality without change of sign. (See Theorem BA-14 above, or Theorem BR-1 and Corollary 3, Section 9–7.)

Hence, to prove the theorem it is sufficient to prove instead that

$$(a \vee b) + ab = a + b.$$

This can be accomplished as follows:

$$
\begin{aligned}
(a \vee b)+ab &= (a \vee b)(ab)' \vee (a \vee b)'(ab) && \text{by Definition BA-1,}\\
&= (a \vee b)(a' \vee b') \vee (a'b')(ab) && \text{by Theorems BA-10}' \text{ and 10,}\\
&= [a(a' \vee b') \vee b(a' \vee b')] \vee Z && \text{by Theorem BA-9}', \text{commutative, associative, complement postulates,}\\
&= (aa' \vee ab') \vee (ba' \vee bb') && \text{by distributive and zero element postulates,}\\
&= (Z \vee ab') \vee (ba' \vee Z) && \text{by complement postulate,}\\
&= ab' \vee a'b && \text{by Theorem BA-2, zero element and commutative multiplication postulates,}\\
&= a+b && \text{by Definition BA-1.}
\end{aligned}
$$

EXERCISE

Given the collection of symbols α, β. Let H be interpreted as the set of all subclasses of these symbols taken two at a time, one at a time, and none at a time. Then H is a (concrete) Boolean algebra under logical sum and logical product, and a (concrete) Boolean ring under symmetric difference and logical product. Designate the classes of H as follows:

$$U = [\alpha, \beta], \qquad B = [\beta],$$
$$A = [\alpha], \qquad Z = [—].$$

The following are sample operation results:

$$A \vee B = U, \qquad A \cdot B = Z \text{ (making } A \text{ and } B \text{ divisors of zero),}$$
$$A + U = B, \qquad A \cdot U = A.$$

Make complete tables for the three operations and also a table of complements. The tables illustrate a Boolean algebra (and Boolean ring) of four elements.

In the same way, starting with three symbols α, β, γ, and using H as the set of all subclasses of these, we can make operation tables for an 8-element Boolean algebra.

9–10 Deriving a Boolean algebra from a Boolean ring with unity. We now continue our list of Boolean ring theorems started in Section 9–7, making the additional assumption that the ring satisfies the unity element postulate (Postulate 11 of Section 9–6 — but since a Boolean ring is commutative we need only require a one-sided unity element). We wish to show that on introducing a suitable definition for Boolean addition, the elements of such a Boolean ring will satisfy the postulates for a Boolean algebra. We previously proved Theo-

rems BR-1 to BR-4, of which Theorem BR-2 was the commutative law of multiplication. We continue here with the same numbering sequence.

In the following, we use B_1 to designate a Boolean ring with unity element. The unity element and zero element will be designated as usual for rings by 1 and 0. These will be found to meet the requirements of the U and Z of the Boolean algebra postulates.

In what follows we shall often dispense with parentheses when three or more terms occur in a sum, or three or more factors in a product. This is permissible as a generalized principle resulting from the associative postulate (as mentioned back in Section 2–6).

We begin with the following *definition* for Boolean addition in a Boolean ring; this is the same in form as Theorem BA-20, proved in the preceding section for Boolean algebras.

DEFINITION BR-1. $a \vee b = (a + b) + ab$, or $a + b + ab$.

THEOREM BR-5. *For every* a, b *in* B_1, $a \vee b$ *belongs to* B_1.

Proof. This follows at once from Definition BR-1 and the ring closure postulates for multiplication and ring addition.

THEOREM BR-6. $a \vee 0 = a$.

Proof. $a \vee 0 = (a + 0) + a0$ by Definition BR-1,

$$\left.\begin{aligned} &= a + 0 \\ &= a \end{aligned}\right\}$$ by zero element ring properties.

THEOREM BR-7. $a \vee b = b \vee a$.

Proof. This follows from the commutative addition property for any ring, and the commutative multiplication property proved in Theorem BR-2.

THEOREM BR-8. $(a \vee b) \vee c = a \vee (b \vee c)$.

Proof. This is left as an exercise.

THEOREM BR-9. $a \vee bc = (a \vee b)(a \vee c)$.

Proof. $a \vee bc = a + bc + abc$ by Definition BR-1.

Also $(a \vee b)(a \vee c) = (a + b + ab)(a + c + ac)$

$= [a + (b + ab)][a + (c + ac)]$

by ring associative addition postulate,

$= aa + (b + ab)a + a(c + ac) + (b + ab)(c + ac)$

by ring right and left distributive postulates,

$=aa+(ba+aba)+(ac+aac)+(bc+bac)+(abc+abac)$
by ring distributive postulates and associative postulates,
$=a+(ba+ab)+(ac+ac)+bc+(abc+abc)+abc$
by idempotent postulate, Theorem BR-2, and associative postulates,
$=a+0+0+bc+0+abc$ by Theorems BR-2, BR-1,
$=a+bc+abc$
by zero element and associative postulates.

$\therefore \quad a \vee bc=(a \vee b)(a \vee c).$

THEOREM BR-10. $a(b \vee c) = ab \vee ac.$

Proof. This is left as an exercise.

DEFINITION BR-2. For every element a in B_1, the element a' is defined to be

$$a' = a + 1.$$

THEOREM BR-11. *For every a in B_1, a' is an element of B_1 such that*

$$a \vee a' = 1 \quad \text{and} \quad aa' = 0.$$

Proof.

a' is an element of B_1 — by addition closure postulate,
$a \vee a' = a \vee (a+1)$ — by Definition BR-2,
$= a + (a+1) + a(a+1)$ — by Definition BR-1,
$= (a+a) + 1 + (aa + a1)$ — by associative addition, left distributive ring postulates,
$= (0+1) + (a+a)$ — by Theorem BR-1, idempotent postulate, and unity element postulate,
$= 1+0$, $= 1$ — by zero element postulate.

Also $aa' = a(a+1)$ — by definition,
$= aa + a1$ — by left distributive postulate,
$= a + a$ — by idempotent postulate and unity element postulate,
$= 0$ — by Theorem BR-1.

THEOREM BR-12. $a + b = ab' \vee a'b.$

Proof. This is left as an exercise.

Note that Theorem BR-12 is the *definition* of $a + b$ in the development based on Boolean algebra postulates, while Definition BR-2 was suggested by Theorem BA-12.

THEOREM BR-13. *With respect to the operations of Boolean addition* ($\vee$) *and multiplication, the elements of any Boolean ring with unity element form a Boolean algebra.*

Proof. The Boolean algebra postulates 1′ and 3′ for multiplication closure and associative multiplication are clearly satisfied, as these hold in any ring.

Commutative multiplication holds by Theorem BR-2. The Boolean addition postulates hold as follows: closure by Theorem BR-5, commutative addition by Theorem BR-7, and associative addition by Theorem BR-8. The distributive postulates 4 and 4′ for Boolean algebras hold by Theorems BR-9 and BR-10. Postulate 5 is satisfied by taking the zero element of the ring as the required element Z. Postulate 5′ is satisfied by taking the unity element of the ring as the required universal element. Postulates 6,6′ hold by Theorem BR-11.

This completes the development which, in conjunction with the development of the preceding section, shows that, on introduction of appropriate definitions, Boolean rings with unity element and Boolean algebras become logically equivalent systems.

9–11 Inclusion in abstract Boolean algebras. We consider finally some abstract definitions and theorems of Boolean algebras concerning the relation of *inclusion*, which we have previously used (in Chapter 4) for classes alone. In this section elements are understood to belong again to a Boolean algebra H.

DEFINITION BA-2. If $a' \vee b = U$, we say that *a is included in b* (or *b* includes *a*), and we write

$$a \subset b \quad (\text{or} \quad b \supset a).$$

We note that if the two-element propositional interpretation of a Boolean algebra is being used, then

$$a' \vee b = U$$

means "not-a or b" has the value "true." Hence, in this case "a is included in b" means essentially "*a implies b* is true."

First we prove a theorem which will be useful for purposes of

dualizing. When we refer to "dual statements" we recall that we mean statements obtainable from each other by interchanging the operations of $\vee$ and $\times$, and interchanging the elements U and Z.

THEOREM BA-21. *The statements $a \subset b$ and $a \supset b$ in a Boolean algebra are dual statements.*

Proof. By definition, $a \subset b$ means $a' \vee b = U$. The dual of the latter statement becomes

$$a' \cdot b = Z,$$

which in turn is equivalent to

$$(a'b)' = Z' = U$$

or	$(a')' \vee b' = U$	by Theorem BA-10′,
or	$a \vee b' = U$	by Theorems BA-5,5′,
or	$b' \vee a = U$	by commutative postulate for Boolean addition.

The latter, by definition, says

$$b \subset a, \quad \text{that is,} \quad a \supset b.$$

By virtue of the theorem just proved we can dualize all of the theorems which follow by changing $\subset$ to $\supset$ and making the other modifications as usual.

THEOREM BA-22. $a \subset a$.

Proof. This follows immediately from the definition and the complement postulate.

THEOREM BA-23. *$a \subset b$ if and only if $ab' = Z$.*

Proof. By definition, $a \subset b$ is equivalent to

$$a' \vee b = U.$$

But this is equivalent to

$$(a' \vee b)' = Z$$

or	$(a')'b' = Z$	by Theorem BA-10,
or	$ab' = Z$	by Theorems BA-5,5′.

THEOREM BA-24. *$a \subset b$ if and only if $ab = a$.*

Proof. "Only if" part. Given $a \subset b$, we show $ab = a$.

Here $ab' = Z$ by Theorem BA-23,
and $ab = ab \vee Z$ by zero element postulate,
$= ab \vee ab'$
$= a(b \vee b')$ by distributive postulate,
$= aU$ by complement postulate,
$= a$ by universal postulate.

"If" part. Given $ab = a$, we show $a \subset b$.

Here $ab' = (ab)b'$
$= a(bb')$ by associative postulate for multiplication,
$= aZ$ by complement postulate,
$= Z$ by Theorem BA-7′.

COROLLARY 1. *For every a in H, $Z \subset a$ and $a \subset U$.*
COROLLARY 2. *If $a \subset b$ and $b \subset a$, then $a = b$.*

THEOREM BA-25. *$a \subset b$ if and only if $a \vee b = b$.*

Proof. This is left as an exercise.

COROLLARY. *$ab = a$ if and only if $a \vee b = b$.*

The next theorem is the abstract analog of the opposite converse equivalence rule for propositions.

THEOREM BA-26. *$a \subset b$ if and only if $b' \subset a'$.*

Proof. Using our usual notation for logical equivalence,

$(a \subset b) \longleftrightarrow (ab' = Z)$ by Theorem BA-23.
$(ab' = Z) \longleftrightarrow [b'(a')' = Z]$ by commutative multiplication postulate and Theorems BA-5,5′.

The statement in brackets, in turn, is logically equivalent to

$b' \subset a'$ by Theorem BA-23.

THEOREM BA-27. *If $a \subset b$ and $b \subset c$ then $a \subset c$.*

Proof. From the given hypotheses, we have both

$ab = a$ and $c = b \vee c$ by Theorems BA-24 and BA-25.
$\therefore$ $ac = a(b \vee c)$
$= ab \vee ac$ by multiplication distributive postulate,
$= a \vee ac$
$= a$ by Theorem BA-8.
$\therefore$ $a \subset c$ by Theorem BA-24.

The preceding theorems are now available for interpretation from the standpoint of either the algebra of propositions or the algebra of classes. We note here that the class interpretation is the typical interpretation in the following sense: *every* (abstract) *Boolean algebra* is isomorphic to a certain (concrete) algebra of classes. This is a theorem due to Stone,* which can be proved by advanced methods of modern algebra.

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* M. H. Stone, "The Theory of Representations for Boolean Algebras," *Transactions of the American Mathematical Society*, Vol. 40, 1936, pp. 37–111.

CHAPTER 10 *

RELATIONS, ORDER SYSTEMS, LATTICES

10–1 Introduction. In this chapter we consider some important mathematical systems which are connected primarily with *relations* rather than with operations. First we discuss relations in general, then equivalence relations, linearly ordered systems, partially ordered systems, and lattices.

10–2 Relations in general. A relation between two elements can be said to exist whenever a specified propositional function R of two variables x, y is satisfied by these two elements. For a given meaning of R we use the notation

$$xRy$$

to mean that x has the relation R to y for specified "values" of x and y.

If R means "is father of" then xRy means "x is father of y." The relation is *satisfied* by *all* pairs and *only* pairs of elements (x, y) for which x designates a male parent and y one of his offspring.

If x and y stand for positive integers and R stands for the relation "greater than," then xRy holds for every pair of positive integers (x, y) having the property that x is greater than y, and otherwise R fails to hold. However, if R stands for "equals," then xRx holds for every pair (x, x), while xRy fails for two unequal values of x and y.

Relations can be described or classified in various standard ways, as indicated in the following definitions. In each case we suppose that the elements concerned belong to a certain fundamental set.

DEFINITION 1. (a) A relation R is said to be *reflexive* on a set S if, *for every* x *in* S, x has relation R to itself, so that xRx is true.

(b) R is *irreflexive* if, *for every* x, xRx is false. (Note that this is stronger than "not reflexive.")

DEFINITION 2. (a) A relation R is *symmetric* on a set S if, whenever xRy holds in S, then yRx holds also; i.e., always xRy *implies* yRx.

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* This chapter may be omitted without loss of continuity.

(b) R is *asymmetric* in S if, *for every* x, y in S, it is false that *both* xRy and yRx. (This is stronger than "not symmetric.")

DEFINITION 3. A relation R is *transitive* on a set S if, *for every* x, y, z in S, the conjunction, xRy and yRz, *implies* xRz.

Consider some illustrations. Suppose R_1 stands for the relation "husband of." Then R_1 is irreflexive and asymmetric: for x is never the husband of x, and it is always false that both x is the husband of y and y is the husband of x. (But we have to admit that R_1 is a transitive relation. This is because the transitivity property holds "vacuously" by virtue of the falsity of its hypothesis, namely the conjunction "x is the husband of y and y is the husband of z"!)

Suppose R_2 is the relation $\leqq$ (less than or equal to) for real numbers. Then R_2 is reflexive (for always $x \leqq x$), and transitive (for $x \leqq y$ and $y \leqq z$ implies $x \leqq z$). But R_2 is *not* symmetric (for $x \leqq y$ does not imply $y \leqq x$); and at the same time it is *not* asymmetric (as it is possible to have both $x \leqq y$ and $y \leqq x$ by simply having x and y equal).

Suppose R_3 is the relation "congruent to" for triangles in geometry (Euclidean or non-Euclidean). Then R_3 is reflexive, symmetric, and transitive.

EXERCISES

For each of the following sets decide, if possible, whether the indicated relation is reflexive, irreflexive, symmetric, asymmetric, transitive:

1. The relation $<$ for real numbers.
2. The relation "sister of" for a family of sisters. (Note that transitivity fails, for "x is sister of y and y is sister of x" does not imply "x is sister of x.")
3. The relation "square root of" for positive integers.
4. The relation "negative of" for real numbers.
5. The relation "ancestor of" for a set of lineal descendants of a certain individual, one in each generation for 10 generations.
6. The relation "is fond of" for persons.
7. The relation "cousin of" for persons.
8. The relation "multiple of" for positive integers.
9. The relation "is a subset of" for a given collection of sets of objects.

10–3 Equivalence relations. We now consider different abstract systems of relations which are of sufficient importance to be designated by special names. It is convenient to use a postulational approach here. We begin with postulates suggested by the usual notion of

equality, which can be generalized to the concept of an *equivalence relation*.

Postulates for Equivalence Relations

Given a set S of elements and a relation R which may or may not hold between pairs of these elements, R is called an *equivalence relation* on S, if the following postulates are satisfied:

Postulate 1. R is reflexive.

Postulate 2. R is symmetric.

Postulate 3. R is transitive.

These postulates are clearly satisfied by equality, in the sense of identity of numbers. They are also satisfied by equality in a more general sense.

For example, in the definitional development of the number system a rational number is defined as a pair of natural numbers a/b ($b \neq 0$), and two rational numbers a/b and c/d are defined to be equal if

$$ad = bc.$$

Assuming the usual properties for natural numbers, we are able to say:

1. $a/b = a/b$ (since $ab = ba$).
2. $a/b = c/d$ implies $c/d = a/b$ (for $ad = bc$ implies $cb = da$).
3. $a/b = c/d$ *and* $c/d = e/f$ implies $a/b = e/f$. (For $ad = bc$ and $cf = de$ implies $adcf = bcde$ which implies $af = bc$. There is a special case here if $a = 0$, but then c and e must also $= 0$, so the conclusion still holds.)

Thus the defined equality for rational numbers is an equivalence relation, just as much as identity of natural numbers. Similar statements hold for equality of real numbers, or of complex numbers.

In the same way, congruence of triangles (or other figures) in geometry is an equivalence relation.

Another type of equivalence relation is the relation "congruent modulo m" which is used in the theory of numbers, and which has been the basis of constructing our modulo systems. To check the postulates here we define the concept officially as follows:

Definition. Given any integers a, b, m (positive, negative or 0, although usually m is a positive integer greater than 1), if $a - b$ is an integral multiple of m we say that a is *congruent* to b *modulo* m and write

$$a \equiv b \bmod m, \quad \text{or} \quad a \equiv b (\bmod m).$$

The following statements show that this is an equivalence relation:

1. $a \equiv a \bmod m$ (since $a - a$ or 0 is a multiple of m).
2. $a \equiv b \bmod m$ implies $b \equiv a \bmod m$ (since if $a - b$ is a multiple of m, then $b - a$ is a multiple of m).
3. $a \equiv b \bmod m$ and $b \equiv c \bmod m$ implies $a \equiv c \bmod m$ (for we have $a - b = km$ for some integer k, $b - c = lm$ for some integer l, whence $a - c = (k + l)m$, so $a - c$ is a multiple of m also).

Next we prove a general theorem which is of considerable significance in the theory of numbers and in abstract algebra.

THEOREM. *An equivalence relation R on a set S separates S into mutually exclusive subsets such that* (1) *all pairs of elements in the same subset are in relation R to each other;* (2) *no two elements in different subsets are in relation R to each other.*

Proof. Take any element t belonging to S. Form the subclass T of S by placing in T all elements x belonging to S such that

$$xRt$$

(and hence tRx by the symmetric property of the equivalence relation R). Of course, the element t itself will be in T (by the reflexive property of R).

If the subclass T is not the whole set S, we take another element u, belonging to S but not to T, and form a subclass U as before, consisting of all elements x satisfying

$$xRu.$$

T and U can have no element in common. For suppose an element y belonged to both T and U, then we would have both

$$yRt \quad \text{and} \quad yRu,$$

that is,

$$yRt \quad \text{and} \quad uRy \quad \text{(by symmetry)}.$$

But by the transitive property this would make

$$uRt,$$

and thus u would belong to the subclass T, contrary to agreement.

If T and U together do not exhaust S, we take another element belonging to S but not to T or U and proceed as before to form a new subclass having no elements in common with T or U. In this way, every element of S belongs to one and only one subset of the type specified, and S is separated as required.

We show now how this abstract theorem can be used within the theory of numbers. First we mention the following fundamental theorem and corollary which are proved in that subject:

THEOREM N.* *Given any two integers a and m with integer $m \neq 0$, then there exist unique integers q and r (designating quotient and remainder) such that*

$$a = q \cdot m + r, \quad \text{where} \quad 0 < r < |m|.$$

COROLLARY 1. *If m is any positive integer, every integer is congruent modulo m to one and only one of the integers* $0, 1, 2, \ldots, m - 1$.

Now, making use of our general theorem about equivalence relations and the fact that "congruent modulo m" is such a relation, we have immediately the following result:

COROLLARY 2. *The relation "congruent modulo m" (where $m > 0$) separates the set of all integers into m mutually exclusive subclasses such that all of the integers in any one subclass are congruent to exactly one of the "remainders"* $0, 1, 2, \ldots, m - 1$, *while no integers in different subclasses are congruent to each other.*

These subclasses are known as *residue classes*, and any integer belonging to a certain residue class is called a representative of that class. The remainders

$$0, 1, 2, \ldots, m - 1$$

can conveniently be used as *representatives* of the different residue classes modulo m.

It is easy to show that if

$$a \equiv a' (\text{mod } m) \quad \text{and} \quad b \equiv b' (\text{mod } m),$$

then

$$a + b \equiv a' + b' (\text{mod } m)$$

and

$$ab \equiv a'b' (\text{mod } m).$$

This indicates that the sum of two integers is congruent to the sum of any representatives of the residue classes to which they belong, and similarly for the product. Thus our familiar addition and multiplication tables modulo m (in which the marginal entries are $0, 1, 2, \ldots, m - 1$) can be thought of as tables for addition or multiplication modulo m of any members of the residue classes having the indicated representatives.

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* For a proof, see Marie Weiss, *Higher Algebra for the Undergraduate*, John Wiley, 1949, p. 13; or G. Birkhoff and S. MacLane, *Survey of Modern Algebra*, Macmillan, 1941, pp. 15–16.

EXERCISES

1. Prove that, in ordinary logic, logical equivalence of propositions is an equivalence relation. Describe how the relation separates the class of all propositions.

2. Prove that the three postulates for an equivalence relation are independent.

3. Prove (by elementary algebra) the rules mentioned above for adding and multiplying congruent integers.

10–4 Linearly ordered systems. In Section 6–7 we proved certain theorems for the relation $>$ which hold within ordered fields. Using $a < b$ here instead of $b > a$, we can easily obtain the following properties for elements of ordered fields:

1. If $a \neq b$, then $a < b$ or $b > a$.

2. If $a < b$, then $a \neq b$, that is, $a < a$ is false. (Irreflexive property for $<$.)

3. If $a < b$ and $b < c$, then $a < c$. (Transitive property for $<$.)

Corresponding to the first property, we make a new abstract definition for a type of relation and then introduce the abstract postulates for *linear order*.

DEFINITION. Given a set of elements and a relation R which may or may not hold between pairs of elements, the relation is *determinate* if $a \neq b$ implies either aRb or bRa.

POSTULATES FOR LINEAR ORDER

A set of elements S is said to be *linearly ordered*, or to form a *linear ordering*, with respect to a relation R if the following postulates hold on S:

POSTULATE 1. R is determinate.

POSTULATE 2. R is irreflexive.

POSTULATE 3. R is transitive.

These three postulates are consistent, since they are satisfied by the $<$ relation, or equally well the $>$ relation, in any ordered field.

Thus the rational numbers or the real numbers (but not the complex numbers) form linearly ordered systems with respect to either $<$ or $>$.

The following theorem is easily proved.

THEOREM. *If aRb holds within a linearly ordered system S, then bRa is false (i.e., R is asymmetric on S).*

EXERCISES

1. Prove the theorem just above.
2. Write out detailed statements for the postulates of linear order.
3. Are the following examples of linearly ordered systems? Why or why not?

(a) The set of all natural numbers 1, 2, 3, . . . with respect to the relation $<$.

(b) The set of all rational numbers with respect to the relation $\geqq$.

(c) The set of all points in an x, y coordinate system with the understanding that any point A is in relation R to point B, if the abscissa of A is less than the abscissa of B (otherwise not).

(d) The set of all points on the x axis with point A in relation R to point B if the abscissa of A is greater than the abscissa of B.

(e) The set of integers 1, 2, 4, 8, 24 with the understanding that xRy when integer x is an exact divisor of integer y.

(f) The set of paternal ancestors (father, paternal grandfather, paternal great grandfather — i.e., paternal grandfather's father — etc.) of a given person, going back five generations, with xRy interpreted as "x is descendant of y." What about the set of *all* ancestors for five generations?

4. Prove that the postulates for linear order are independent.

10–5 Partially ordered systems. A type of order more general than linear order is *partial order*. It is more general in essentially two different ways: (a) the determinate property for relation R is not required for a partially ordered system; (b) a typical linear ordering relation is the relation $<$ (or $>$) as used for real numbers, while a typical partial ordering relation is the less restricted relation $\leqq$ (or $\geqq$).

Before listing the postulates for partial order, we introduce a definition for a new type of relation property.

DEFINITION. A relation R is called *antisymmetric* on a set S if, *for every* x, y belonging to S, the conjunction xRy and yRx *implies* $x = y$.

Note for comparison that the *symmetric* property says that *always* xRy implies yRx; the *asymmetric* property says that xRy and yRx *never* hold at the same time; while the *antisymmetric* property says that xRy and yRx hold simultaneously *only if* $x = y$. Now we proceed to the postulational definition for partial order.

POSTULATES FOR PARTIAL ORDER

A set of elements S is said to be *partially ordered*, or to form a *partial ordering* with respect to a relation R, if the following postulates hold on S:

POSTULATE 1. R is reflexive.
POSTULATE 2. R is antisymmetric.
POSTULATE 3. R is transitive.

A simple illustration of a partially ordered system is the set of all natural numbers under the relation $\leqq$ or $\geqq$, as we have already suggested. This is because: (1) we always have $x \leqq x$; (2) $x \leqq y$ and $y \leqq x$ *implies* $x = y$; (3) $x \leqq y$ and $y \leqq z$ *implies* $x \leqq z$.

Another illustration of more interest is the set S of all subclasses of a certain class of objects, under the relation "is subclass of," or "is contained in."

We repeat the definition of Chapter 4 that one class X is contained in another class Y if every member of X is also a member of Y. In this case we write $X \subset Y$. Also, two classes are understood to be *equal* when they are identical in their membership.

To check the postulates, we merely note the following properties, which are an immediate result of the definition:

(1) For every class X belonging to S, $X \subset X$.

(2) For every X, Y in S, if $X \subset Y$ and $Y \subset X$, then $X = Y$. (This is because every member of each class belongs to the other, so the classes have the same membership.)

(3) If $W \subset X$ and $X \subset Y$, then $W \subset Y$.

As a simple special case let the set S be the set of all subclasses of the class I consisting of the two letters α and β (cf. the exercise of Section 9–9). From the standpoint of college algebra these subclasses can be described as the *combinations* of the given letters taken *two at a time*, *one at a time*, and *none at a time*. Thus set S comprises the following classes of letters:

I: $[\alpha, \beta]$ $\quad$ B: $[\beta]$
A: $[\alpha]$ $\quad$ Z: $[—]$

It is understood always that the "empty class," Z, is contained in every class (including itself). Accordingly, for distinct classes we have the following relations:

$$A \subset I, \qquad B \subset I,$$
$$Z \subset A, \qquad Z \subset B, \qquad Z \subset I.$$

But note that A is not contained in B, nor is B contained in A. This can be written:

$$A \not\subset B, \qquad B \not\subset A.$$

In other words, A and B are noncomparable, which indicates more of a deviation from linearly ordered systems than in the case of real numbers under the relation $\leqq$.

The results can be summarized in the following table, in which we place a check mark in the appropriate position if and only if one class is contained in another, reading from left to right in the familiar way:

$\subset$	Z	A	B	I
Z	✓	✓	✓	✓
A		✓		✓
B			✓	✓
I				✓

Exercises

1. List the classes forming the partially ordered system consisting of all subclasses of the set of three letters a, b, c, and prepare a table showing which classes are contained in others.

2. Are the following examples of partially ordered systems?

(a) The set of integers

1, 2, 3, 4, 6, 12

with xRy interpreted to mean (i) "x is an exact divisor of y," (ii) "x is a multiple of y."

(b) The set of all descendants of a given person, together with the person himself, interpreting xRy to mean "x is the same person as y" *or* "x is a descendant of y."

(c) The same as (b) except that xRy is to mean "x is in the same generation as y."

3. In a partially ordered system S (under relation R) define xCy to mean "xRy but $x \neq y$." Show that if relation C is determinate, S becomes linearly ordered with respect to C. Illustrate.

4. In a linearly ordered system T (under relation Q) define xPy to mean

"xQy or $x = y$." Prove that T becomes partially ordered under relation P. Illustrate.

5. Illustrate partially ordered systems by classification charts of families, species, etc. in botany or zoology.

10–6 Duality in partially ordered systems. In this section we return to the concept of duality in a new form. Duality in partially ordered systems actually has connections with duality in projective geometry and Boolean algebras, through the theory of lattices.

DEFINITION. In a system S which is partially ordered under relation R, let R' be the relation between y and x which holds whenever relation R holds between x and y, so that

$$xRy \text{ is equivalent to } yR'x.$$

Then R' is called the *dual* relation to R.*

For example, if R is the relation $\leqq$ for real numbers, then R' is $\geqq$. Or if R is the relation "is exact divisor of" for integers, then R' is the relation "is multiple of." If R is the relation "cousin of" for persons, then R' is also "cousin of," and similarly whenever R is a symmetric relation.

THEOREM. Duality principle for partially ordered systems. *Any system S which is partially ordered under relation R is likewise partially ordered under the dual relation R'.*

Proof. From the postulates for partial order, for every x, y, z in S we have, using our symbolic logic notation,

$$\begin{aligned} &xRx, \\ &(xRy) \cdot (yRx) \rightarrow x = y, \\ &(xRy) \cdot (yRz) \rightarrow xRz. \end{aligned}$$

Changing each of these to statements concerning relation R', we obtain

$$\begin{aligned} &xR'x, \\ &(yR'x) \cdot (xR'y) \rightarrow x = y, \\ &(zR'y) \cdot (yR'x) \rightarrow zR'x. \end{aligned}$$

But these statements tell us that R' is a relation on S which also satisfies the partial order postulates. Hence S is also a partially ordered system under R'.

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* More generally, from the standpoint of relations, R' is called the *converse* relation of R.

10–7 Lattices. Systems known as lattices are important special types of partially ordered systems. In recent years, through the research of Garrett Birkhoff * and others, lattice theory has been shown to have considerable unifying influence on many apparently distinct branches of mathematics (including modern algebra, topology, theory of sets, projective geometry, logic).

In order to give the definition of a lattice we must first introduce the notion of abstract *meet* and *join* in a partially ordered system.

DEFINITION 1. If x, y are elements (distinct or not) belonging to a partially ordered system S, and g is an element of S such that both gRx and gRy, then g is called a *lower bound* of x and y.

If for any lower bound, g', of x and y, $g'Rg$ holds, then g is called a *greatest lower bound* (GLB), or *meet*, of x and y.

DEFINITION 2. If x, y are elements (distinct or not) belonging to a partially ordered system S, and h is an element of S such that both xRh and yRh, then h is called an *upper bound* of x and y.

If for any upper bound, h', of x and y, hRh' holds, then h is called a *least upper bound* (LUB), or *join*, of x and y.

THEOREM 1. *If two elements x and y in a partially ordered system S have a meet (or join), the meet (or join) is unique.*

Proof. Suppose g_1 and g_2 are both meets of x and y. By definition, g_1 and g_2 belong to S, and we have both

$$g_1Rg_2 \quad \text{and} \quad g_2Rg_1,$$

whence $g_1 = g_2$ by the antisymmetric property of relation R in S.

A similar proof holds for joins.

COROLLARY. *Two elements each equal to x in a partially ordered system S always have x itself as both meet and join.*

The notion of meet and join can easily be extended to the case of a finite set of elements.

Next we come to the definition of a lattice.

DEFINITION 3. A *lattice* is a partially ordered system S in which every pair of elements x, y has a meet and a join.

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* See G. Birkhoff, *Lattice Theory*, American Mathematical Society, 1940; also G. Birkhoff and S. MacLane, *Survey of Modern Algebra*, Macmillan, 1941, Chapter XI.

The meet of x and y is commonly designated as $x \cap y$ and the join as $x \cup y$.* It is easy to use the duality principle of partially ordered systems to prove a corresponding principle (Theorem 3 below) for meets and joins in lattices. But first we prove the following.

THEOREM 2. (a) *If g is the meet, $x \cap y$, of x and y in a partially ordered system S with respect to the fundamental relation R of the system, then g is the join of x and y with respect to the dual relation R'.*

(b) *If u is the join, $x \cup y$, of x and y with respect to relation R, then u is the meet of x and y under relation R'.*

Proof. (a) We have gRx, gRy, and if also hRx, hRy, then hRg by the GLB property.

Hence $xR'g$, $yR'g$, and if also $xR'h$, $yR'h$, then $gR'h$, so g satisfies the definition of *join* of x and y for the relation R'.

A similar proof holds for (b).

THEOREM 3. Duality principle for lattices. *Any theorem concerning meets and joins in an arbitrary lattice must hold equally well when the words "meet" and "join" are interchanged.*

Proof. By Theorem 2, any partially ordered system which is a lattice under relation R is also a lattice under the dual relation R'. A statement proved about meets and joins with reference to relation R would yield, by the same argument, the same statement about meets and joins under relation R'. But this could be translated back into a statement about joins and meets, respectively, under the original relation R.

The following are illustrations of lattices.

Illustration 1. The partially ordered system of all subsets (including the empty set) of a given class I forms a lattice under the relation of set-inclusion ($\subset$). To justify this statement, we can show that if X, Y are any two subsets of the system, then their logical product XY (i.e., the class of all members common to both X and Y) serves as their meet.

This is because

$$XY \subset X, \quad XY \subset Y,$$

and if W is any set such that

$$W \subset X, \quad W \subset Y,$$

---

* These can be read respectively as "x meet y" and "x join y."

then automatically we have $W \subset XY$. Thus XY satisfies the requirements for the abstract meet of X, Y.

Similarly, the union or logical sum of X, Y satisfies the requirements for the join of X and Y.

This lattice has a *universal* upper bound, namely, the whole class I. This means that every "element" (i.e., subset) of the system is contained in the class I. Also the empty set is a universal lower bound.

Illustration 2. The system of all positive integers under the relation "divides" forms a lattice. By "m divides n" we mean integer m is an *exact divisor* of integer n. It is readily seen that the system is partially ordered. The meet of any two integers m and n is simply their *highest common factor*. This is because, by definition, the HCF of m and n divides m and n, and also any other common factor divides the HCF. In the same way, the join of any two integers is their *least common multiple*.

This lattice has a universal lower bound, namely, the integer 1, but no universal upper bound.

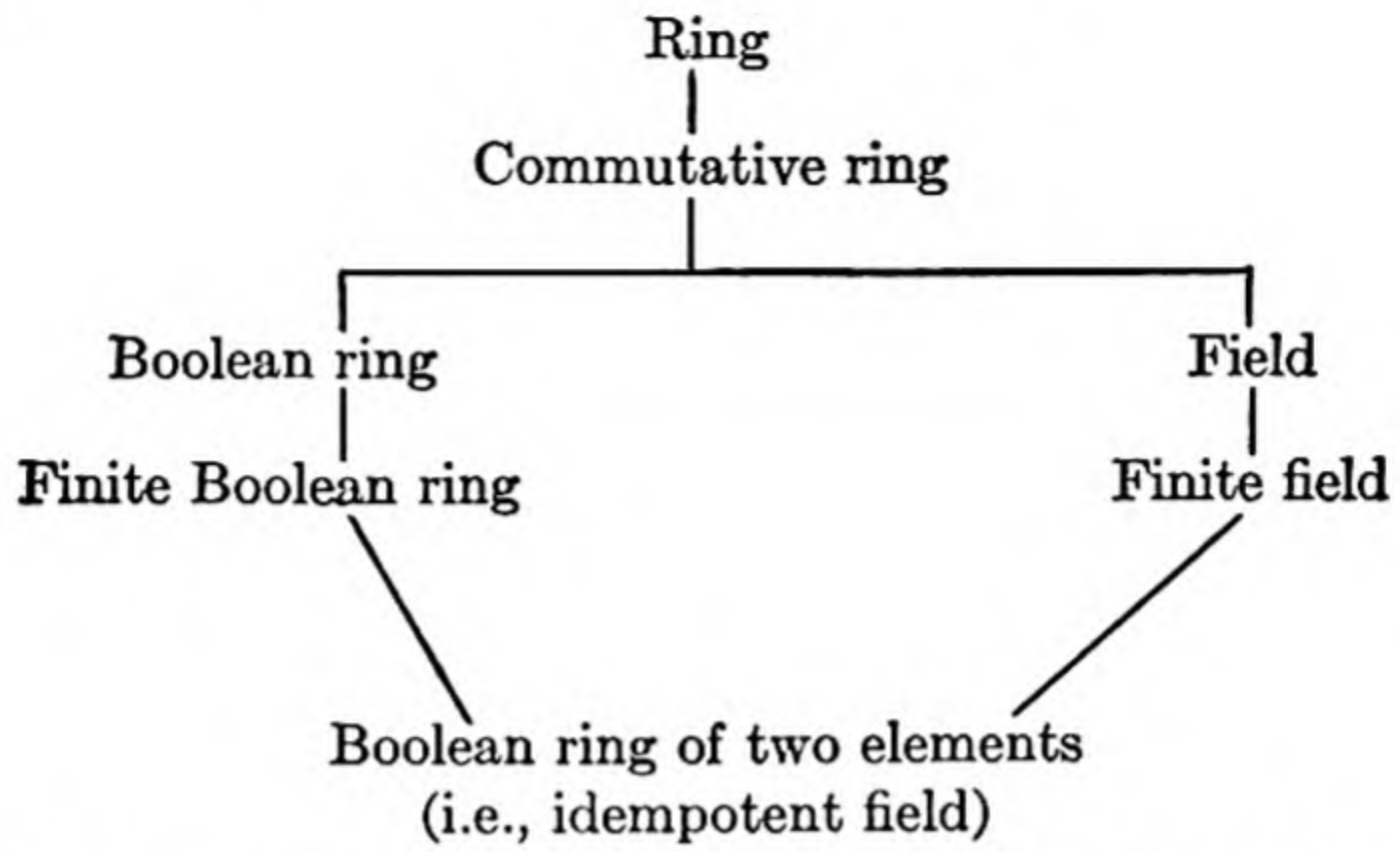
Illustration 3. The system of all real numbers under the relation $\leqq$ forms a lattice. The meet of two real numbers x and y will be the smaller of the numbers (or their common value if they are equal), while the join of x and y will be the greater of the two (or their common value if equal). This lattice has neither a universal upper nor universal lower bound.

Illustration 4. The set of all subgroups of a given group forms a lattice under the relation of set-inclusion. The whole group itself is a universal upper bound and the identity element is a universal lower bound. (For complete details somewhat more group theory is needed than we have covered in Chapter 9.)

Exercises

1. Prove that the following set of integers forms a lattice under the relation "divides": 1,2,3,4,6,8,12,24. Does the lattice have universal bounds?
2. Does the set of *all* ancestors of a given person for three generations form a lattice under the relation "is a descendant of or is the same person as"? Why or why not?
3. Prove the corollary of Theorem 1.
4. (a) Prove that the following genealogical classification of various alge-

braic systems is a lattice under the relation "is specialization of or same system as":



(b) Would there still be a lattice if "Boolean ring of two elements" were removed from the classification?

10–8 Lattices and Boolean algebras. Inasmuch as an abstract Boolean algebra is isomorphic to some algebra of sets (see Section 9–11), it is natural to expect a close relationship between Boolean algebras and lattices. We conclude the chapter by investigating this relationship.

First, we prove the following Boolean algebra theorem (continuing the numbering of Chapter 9).

THEOREM BA-28. *Every Boolean algebra H becomes a lattice under the abstract inclusion relation ($\subset$), with the meet of two elements x, y, belonging to H, equal to xy, and the join equal to $x \vee y$.*

Proof. I. H is a partially ordered system under the relation $\subset$. To prove this, we refer to certain theorems of Section 9–11 which establish the reflexive, antisymmetric, and transitive properties:

(1) $x \subset x$ by Theorem BA-22.
(2) $(x \subset y) \cdot (y \subset x) \rightarrow x = y$ by Corollary 2 of Theorem BA-24.
(3) $(x \subset y) \cdot (y \subset z) \rightarrow x \subset z$ by Theorem BA-27.

II. xy, which belongs to H by closure, is a lower bound for x and y. This is because

$$\begin{aligned} (xy)x &= x(xy) && \text{by commutative postulate,} \\ &= (xx)y && \text{by associative postulate,} \\ &= xy && \text{by idempotent property.} \end{aligned}$$

Hence $xy \subset x$ by Theorem BA-24.
Similarly $xy \subset y$.

III. Now letting w be any lower bound of x and y, we show that w is included in xy:

Since	$w \subset x$ and $w \subset y$,	
we have	$wx = w, \quad wy = w$	by Theorem BA-24,
whence	$(wx)(wy) = w \cdot w = w$	by idempotent property,
or	$(ww)(wy) = w$	by associative and commutative postulates,
and	$w(xy) = w$	by idempotent property.
$\therefore$	$w \subset xy$	by Theorem BA-24.

This shows that xy is the GLB or meet of x and y.

A dual Boolean algebra proof shows that $x \vee y$ is the LUB or join of x and y. Thus H is a lattice under the inclusion relation.

Although this theorem shows that every Boolean algebra can be considered to be a lattice, the converse is not true. For one thing, the postulates for a Boolean algebra require existence of zero and universal elements (universal bounds from the lattice standpoint), and complements, which do not exist in every lattice.

In order to find out just what special lattice properties will suffice to make a lattice a Boolean algebra, we prove next a series of lattice theorems (designated by the label "L").

Lattice Theorems

Theorem L-1. *In any lattice the commutative law holds for meets and joins:*

$$a \cap b = b \cap a \quad \textit{and} \quad a \cup b = b \cup a.$$

Proof. The meet, or GLB, of a and b is the same as the GLB of b and a, by definition, and dually for the joins.

Theorem L-2. *In any lattice the associative laws hold for meets and joins:*

$$(a \cap b) \cap c = a \cap (b \cap c) \quad \textit{and} \quad (a \cup b) \cup c = a \cup (b \cup c).$$

Proof. (For meets.)

Let $g = (a \cap b) \cap c =$ meet of $a \cap b$ and c.		
Hence	$gR(a \cap b), \ gRc,$	where R is the fundamental relation of the lattice.
But	$(a \cap b)Ra$ and $(a \cap b)Rb.$	
$\therefore$	$gRa, \ gRb, \ gRc.$	

In other words, g is a lower bound of a, b, and c.

If x is any lower bound of a, b, c

then $xR(a \cap b)$ by definition of $a \cap b$
and xRc,
whence $xR[(a \cap b) \cap c]$ by definition.
That is, xRg.
Thus g is a GLB of a, b, c.

In the same way, if we put

$$h = a \cap (b \cap c),$$

we find that h is also a GLB of a, b, c.

But by the same argument as in Theorem 1 of the preceding section, the GLB of a, b, c must be unique.

$$\therefore\ g = h, \quad \text{as required.}$$

By duality, the associative law holds for joins.

THEOREM L-3. *In any lattice the following dual "absorption" laws hold:*
(1) $a \cup (a \cap b) = a$, *and* (2) $a \cap (a \cup b) = a$.

Proof of (1). We know that

$$(a \cap b)Ra$$

and $$aRa.$$

Hence a is an upper bound to both a and $a \cap b$. Furthermore, if x is any upper bound of these two elements, we must have

$$aRx,$$

so that a is the least upper bound of a and $a \cap b$, that is,

$$a \cup (a \cap b) = a.$$

Part (2) of the theorem holds by the duality principle.

The analogs of the two dual distributive postulates of Boolean algebras do not hold in every lattice, i.e., these postulates are independent of the lattice postulates. This can be shown as usual by examples. However, if one of the lattice distributive postulates holds, we prove now that the other holds also.

THEOREM L-4. *If in a lattice the following distributive law holds:*

$$x \cap (y \cup z) = (x \cap y) \cup (x \cap z),$$

then the dual distributive law holds also:

$$a \cup (b \cap c) = (a \cup b) \cap (a \cup c).$$

Proof. Expanding the right side of the desired law, using the law given by hypothesis, we show that this equals the left side:

$$\begin{aligned}(a \cup b) \cap (a \cup c) &= [(a \cup b) \cap a] \cup [(a \cup b) \cap c] \\ &= [a \cap (a \cup b)] \cup [c \cap (a \cup b)] \\ &\qquad \text{by commutative property for meets,} \\ &= a \cup [c \cap (a \cup b)] \\ &\qquad \text{by absorption property (2) of Theorem L-3,} \\ &= a \cup [(c \cap a) \cup (c \cap b)] \\ &\qquad \text{by hypothesis,} \\ &= [a \cup (c \cap a)] \cup (c \cap b) \\ &\qquad \text{by associative property for joins,} \\ &= a \cup (c \cap b) \quad \text{by absorption property (1),} \\ &= a \cup (b \cap c) \quad \text{by commutative property.}\end{aligned}$$

DEFINITION L-1. A lattice which satisfies the distributive law $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$ is called a *distributive* lattice.

THEOREM L-5. *If a lattice L contains universal bounds, i.e., elements Z and I such that for all x belonging to L we have*

$$ZRx \quad and \quad xRI,$$

then for every a in L we have

$$a \cup Z = a \quad and \quad a \cap I = a.$$

Proof. This is obvious at once from the definition of meet and join.

DEFINITION L-2. Given a lattice L with universal bounds Z and I. If for every a belonging to L there exists an a' in L such that

$$a \cup a' = I, \quad a \cap a' = Z,$$

then L is called a *complemented* lattice.

We are now in a position to state our sufficient condition for a lattice to be a Boolean algebra.

THEOREM L-6. *If a lattice L is both distributive and complemented, it becomes a Boolean algebra by taking the Boolean sum and product of any two elements x, y of L to be their lattice join and meet respectively, that is,*

$$x \vee y = x \cup y \quad and \quad xy = x \cap y.$$

Proof. Referring to our list of postulates for a Boolean algebra (Section 9–8), we can check off all the postulates as follows:

Closure postulates (1,1′) hold by the lattice property (requiring meets and joins of elements in L to belong to L).
Commutative postulates (2,2′) hold by Theorem L-1 above.
Associative postulates (3,3′) hold by Theorem L-2.
Distributive postulates (4,4′) hold by Definition L-1 and Theorem L-4.
Zero and universal postulates (5,5′) hold by Definition L-2 and Theorem L-5.
Complement postulates (6,6′) hold by Definition L-2.

The above theorem concludes the development of this chapter. It furnishes a good illustration of the unifying power of the abstract postulational approach in mathematics. For we see that Boolean algebras which were shown in Chapter 9 to be closely identified with special types of closed operation systems (rings) are also closely identified with special types of order systems (lattices).

CHAPTER 11

SURVEY OF MATHEMATICAL FOUNDATIONS

11–1 Introduction. In this final chapter we propose to do the following: (1) outline some important postulational investigations which have been carried out in the past for various standard branches of mathematics, including Euclidean and non-Euclidean geometry, the algebra of complex numbers, and the theory of natural numbers; (2) suggest the nature of the foundations of logic and arithmetic presented in the *Principia Mathematica;* (3) survey briefly some other approaches to the foundations of mathematics, especially those of the *formalist* and *intuitionist* schools of thought; (4) indicate the broader significance of various foundational points of view in connection with the nature of mathematical knowledge.

We begin with postulational methods in non-Euclidean geometry. We have already discussed this kind of geometry at some length as an example of a branch of mathematics which historically had to struggle to establish its claim to intellectual respectability. The struggle itself had beneficial by-products leading to more thorough postulational investigations in geometry and other branches of mathematics, and also to a better appreciation of the potential usefulness of somewhat imaginative branches of pure mathematics in physical and astronomical science.

11–2 Postulational methods in non-Euclidean geometry. We shall restrict ourselves here to a sample consistency proof for the postulates of Riemannian geometry, in particular, for a special case of Riemannian geometry known as "double elliptic geometry."

We recall that in Riemannian geometry the Euclidean parallel postulate is replaced by one which reads, "through a point not on a given line there is no line parallel to the given line." At the same time, certain tacit assumptions made by Euclid have to be changed, notably the assumptions that (a) a straight line is infinite in length, and (b) there is *only* one line (i.e., not more than one line) through two given points. In place of these, the Riemannian postulates are: (a′) a straight line is unbounded (and may turn out to be finite in length), and (b′) there is at least one line through two given points.

The desired consistency proof can be supplied in the usual way, by exhibiting what we believe to be a consistent *model* of the Riemannian postulates.

For this purpose let us consider a sphere in Euclidean three-dimensional space. It is a familiar Euclidean property (more strictly speaking, a theorem in calculus of variations) that the shortest path on the surface of a sphere, between two points on the surface, is along the great circle joining these points.* Thus a great circle on a sphere is in this respect analogous to a straight line in a Euclidean plane. But in other respects the analogy breaks down. For example: (1) any two great circles on a sphere must intersect in two points on the sphere, (2) all great circles on a given sphere have a fixed finite length, (3) all great circles at right angles to a given great circle intersect in two points at a distance from it equal to one-quarter of the length of a great circle. These properties are quite different from those of straight lines in the Euclidean plane. Yet they are in abstract agreement with the properties of straight lines in the double elliptic version of Riemannian plane geometry.

To clarify the situation, let us consider "point," "line," and "plane" as undefined terms in the latter geometry. Assuming prior knowledge of Euclidean spherical geometry, we now interpret these terms in accordance with the following glossary:

Riemannian "plane": *surface* S of a fixed Euclidean sphere.
Riemannian "point": *point* belonging to S.
Riemannian "line": *great circle* belonging to S.

We see that with this interpretation the key Riemannian postulate of nonexistence of parallel lines is satisfied. Also the special Riemannian postulates (a′) and (b′) above (of unboundedness of straight lines and existence of *at least one* line through two points) are satisfied. Thus we have exhibited a model for Riemannian geometry within the realm of Euclidean geometry.

Note that we are not claiming by any means that a Riemannian "plane" *is the same as* a spherical surface, or that a Riemannian "line" *is* a great circle. All we are really claiming is that points and great circles on the surface of a sphere in Euclidean geometry

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* A great circle on a sphere is a circle with radius equal to the radius of the sphere, and whose center is the center of the sphere.

have the same logical relationships as the undefined "points" and "lines" in the double elliptic Riemannian "plane." *

More explicitly, we find that in the double elliptic variety of Riemannian plane geometry the following statements hold: two straight lines intersect in exactly two points; all lines perpendicular to a given straight line intersect in two points at a distance from it equal to one-quarter of the total length of a straight line; through certain pairs of points there go infinitely many lines; and, of course, the sum of the angles of a triangle is greater than 180°. The latter corresponds to the familiar Euclidean theorem that the sum of the angles of a spherical triangle (i.e., a triangle whose sides are arcs of great circles on a sphere) is greater than 180°.

As a result of our Euclidean model of Riemannian plane geometry, we now assert that Riemannian plane geometry is entitled to be rated just as consistent as Euclidean geometry. Otherwise any contradiction resulting from the assumed properties of points and lines in Riemannian plane geometry could be translated into a corresponding contradiction in Euclidean spherical geometry.

This is a good illustration of the nonabsoluteness of a consistency proof, because of shifting the burden of proof from one department to another.

The logical consistency of Lobachevskian geometry can be supported in a similar way by finding a Euclidean model for the Lobachevskian postulates. One well-known model for this purpose was suggested by Henri Poincaré, the famous French mathematician and scientific philosopher.

Exercises

1. Investigate Poincaré's model for Lobachevskian geometry. (See John W. Young, *Fundamental Concepts of Algebra and Geometry*, Macmillan, 1925, pp. 15–21. For somewhat simpler models, see Richardson, Chapter XVI; Cooley, Gans, first edition, Chapter XVIII; or R. Courant and H. Robbins, *What is Mathematics?*, Oxford University Press, 1941, pp. 218–224.)

2. Explain the significance of the consistency of the non-Euclidean geometries from the standpoint of investigating the *independence* of Euclid's fifth postulate in relation to his other postulates.

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* We mentioned in Chapter 1 another possible variety of Riemannian plane geometry known as "single elliptic" geometry. In this geometry two lines always intersect in exactly one point, and two lines perpendicular to a given line l intersect in a point which is at a distance from l equal to one-half the total length of a straight line.

We recall that Euclid proved the existence of *one* parallel to a line through a given point by use of his postulates and implicit assumptions. Furthermore, in the presence of the other postulates and tacit assumptions his fifth postulate was equivalent to the assumption of *only one* (i.e., *not more than one*) parallel to a line through a given point. Hence, in order to get a satisfactory proof of independence of the fifth postulate in relation to Euclid's other postulates and tacit assumptions, which system of non-Euclidean geometry should be cited?

11–3 Hilbert's postulates for Euclidean geometry. Turning now to Euclidean geometry, we recall from Chapter 2 that Euclid's postulational foundations for the *Elements* were deficient in various respects. In particular, there were no official undefined terms, and he used tacit assumptions and vague concepts requiring definition in many of his proofs. Furthermore, we can add now that he made no explicit attempt to investigate such things as the consistency, independence, and categoricalness of his postulates. He did seem to have some kind of feeling for independence, as is shown by the fact that he explicitly listed his fifth postulate, instead of attempting to prove it on a makeshift basis or adopting it (or something equivalent) as a tacit assumption. However, he should have gone further in this direction by making official all the other assumptions he needed.

Numerous modern investigations into the foundations of geometry have been made with a view to remedying the defects in Euclid. We have already mentioned those by Hilbert and Veblen.* Here we shall give a brief account of Hilbert's postulates.

Hilbert used *point*, *line*, *plane*, *between*, and *congruent* as undefined terms. He listed six groups of postulates which he called "axioms" of *connection*, *order*, *parallels*, *congruence*, *continuity*, and *completeness*, respectively. In Group 1 we find such postulates as the following:

> Two distinct points determine a straight line.
> Any two points of a straight line determine that line.
> Three noncollinear points determine a plane.
> If two points of a line are in a plane, every point of the line is in the plane.

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* David Hilbert, *Foundations of Geometry*, Open Court Publishing Co., 1921; Oswald Veblen, "A System of Axioms for Geometry," *Transactions of the American Mathematical Society*, Vol. 5, 1904, pp. 343–384; also his monograph "The Foundations of Geometry," in *Monographs on Topics of Modern Mathematics*, edited by J. W. A. Young, Longmans, Green and Co., 1927, pp. 3–51.

Every line contains at least two points; every plane at least three noncollinear points; and space at least four points not in a plane.

The postulates of Group 2 serve to characterize the undefined concept of "betweenness." For example:

If A, B, C are points of a straight line and B lies between A and C, then B lies also between C and A.

Given two points of a straight line, there exists at least one other point of the line which lies between them.

Of three given collinear points, one and only one is between the other two.

Given three points A, B, C which are not on the same straight line, and a straight line in the plane of ABC not passing through any of the points A, B, or C, then if the line contains a point of the segment AB it also contains a point either of the segment BC or of the segment AC.

From the postulates of the first two groups it is possible to prove some of Euclid's tacit assumptions and form definitions corresponding to some of his vague concepts. The following are sample theorems of this type, given by Hilbert:

Every straight line a in a plane divides the remaining points of this plane into two regions having the following properties: Every point A of the one region determines with each point B of the other region a segment AB containing a point of the straight line a. On the other hand, any two points A, A' of the same region determine a segment AA' containing no point of a.

Every simple polygon whose vertices lie in a plane divides the points of this plane not belonging to the broken line constituting the sides of the polygon, into two regions, an interior and an exterior (whose properties are then described).

The *line segment* AB is defined to consist of points A and B together with the points *between* A and B.

Hilbert's Group 3 contains simply what we have called the "Euclidean parallel postulate" (not Euclid's fifth postulate), assuming *both* the existence and uniqueness of a parallel to a given line through an outside point.

Group 4 is concerned with the undefined concept of *congruence* as

used in relation to segments, angles, and triangles, including the following:

> If a segment AB is congruent ($\equiv$) to the segment $A'B'$ and also to the segment $A''B''$, then segment $A'B'$ is congruent to segment $A''B''$.
>
> If in the two triangles ABC and $A'B'C'$ the congruences $AB \equiv A'B'$, $AC \equiv A'C'$, angle $A \equiv$ angle A' hold, then the congruences angle $B \equiv$ angle B' and angle $C \equiv$ angle C' also hold.

The postulate just stated is essentially the familiar proposition of high school geometry that two triangles are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other. Using this as a postulate does away with the need for superposition, or "moving" a figure from one position to another.

In Group 5 we find only the following postulate of continuity, usually named after Archimedes. It also provides for the possibility of measuring a given line segment in terms of a unit line segment:

> *Archimedean postulate* (or assumption of measurement). Let A_1 be any point on a straight line between arbitrary points A and B. Take the points A_2, A_3, A_4, . . . so that A_2 lies between A_1 and A_3, A_3 between A_2 and A_4, etc., and let the segments AA_1, A_1A_2, A_2A_3, . . . be congruent to each other. Then in this succession of points, there exists a point A_n such that B lies between A and A_n.

The last group (not contained in the first version of his postulates) consists of what Hilbert called the postulate of completeness. We might better refer to it as a postulate of *linear* completeness, inasmuch as the word "completeness" is used in a different sense from our concept of completeness of a set of postulates. It is as follows:

> To a system of points, lines, and planes it is impossible to add other elements in such a manner that the system thus generalized shall form a new geometry obeying all of the five groups of axioms.

Hilbert established, or at least supported, the consistency of his postulates by constructing an algebraic model of the postulates.

This model, if we restrict ourselves to an adaptation to two dimensions, is comparable to the interpretation of the undefined concept "point" as a pair of real numbers (x,y), and of a "line" as a linear equation

$$ax + by + c = 0,$$

where the a, b, c coefficients are real numbers. A point is understood to lie on a given line if and only if the coordinates of the point satisfy the equation of the line. Two lines meet or have a point in common if and only if the two equations have a solution. Appropriate meanings are assigned likewise to "between" and "congruent." For example, a line segment AB determined by points (x_1, y_1) and (x_2, y_2) is interpreted as congruent to segment $A'B'$ determined by (x_1', y_1') and (x_2', y_2') if and only if

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = (x_2' - x_1')^2 + (y_2' - y_1')^2.$$

On the basis of interpretations of this kind, Hilbert claimed that all of his postulates were satisfied. Furthermore, he asserted with some supporting examples that the postulates are independent within groups and between groups; but he did not prove that all of the postulates, thrown together in one large set, are independent.

Hilbert did not treat the question of categoricalness of his postulates very explicitly. However, he indicated that his postulate of linear completeness permits establishment of a one-to-one correspondence between the points on a line and the numbers of the real number system. This enables numerical coordinates to be introduced for points of a plane or of space. In this way an isomorphism can be established between any two interpretations satisfying all of the postulates. The resulting categoricalness of his postulates justifies the claim that no further assumptions are needed for clear-cut determination of Euclidean geometric properties.

We note that the consistency (and categoricalness) of Hilbert's postulates hinges very definitely on the consistency of the real number system (and the same statement would hold for Veblen's postulates, or any other formulation of postulates for Euclidean geometry). It will be important, then, for us to take up the question of the consistency of the real number system; we shall do this shortly as part of the wider question of postulates for the algebra of complex numbers. But first we devote one section to a general discussion of the relationship of geometry and algebra from the standpoint of the foundations of each subject.

11–4 Relations of geometry and algebra. Bearings on analytic and projective geometry, and the calculus. The close connection between the domain of space and number indicated in the preceding section is of great significance for the rather loosely organized branch of mathematical study known as *analytic geometry*.

From the geometric foundational standpoint "analytic geometry" (as usually understood) can be thought of as a logical outgrowth of Euclidean geometry, after reaching a point in the development from the postulates of the latter which permits the introduction of a coordinate system for all the points of a plane (or space). Standard Euclidean theorems, such as theorems of congruence, parallels, similarity, Pythagorean theorem, and related methods of trigonometry, are utilized in conjunction with a presupposed knowledge of the algebra of real (occasionally complex) numbers.*

In this way it is possible to arrive at formulas for the distance between two points or the angle between two lines, and to prove that every straight line in the plane has a corresponding *equation* (a linear or first degree equation) connecting the coordinates of a variable point on the line. Similarly, every circle has a special type of second degree equation. These methods also make it possible to go beyond the figures involving straight lines and circles, which are the concern of Euclidean plane geometry, and study other curves, including ellipses, parabolas, hyperbolas, cycloids, and "polar" curves, such as cardioids, spirals, and various curves with "loops" or "leaves." For many of these curves it is possible to start with a geometric definition of the curve as a locus and develop an equation that can be used in conjunction with algebraic or trigonometric methods to obtain further geometric properties of the curve. The algebraic methods amount to a procedure for thinking through a geometric problem using different mental images in a different language, and then translating back the results of the thinking into the original geometric form.

It is this kind of translation process for solving a geometric problem through the medium of analytic geometry and algebraic methods which has permitted the formulation of algebraic criteria for constructibility with rulers and compasses. Since ruler-and-compass constructions involve the use of intersecting straight lines and circles, a proposed construction is possible if and only if it is possible to

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* In this section and the following one, the word "algebra" is used primarily in the traditional rather than the abstract, modern sense.

determine the desired geometric elements analytically from the given geometric elements by combining linear and second degree equations in two variables. This, in turn, shows that a required construction is possible only if the numbers corresponding to the desired elements satisfy an irreducible algebraic equation whose degree is a power of 2. In the case of the famous problems of duplicating the cube, trisecting an *arbitrary* angle, and squaring the circle, it can be shown that this necessary condition fails to be satisfied.*

The methods of analytic geometry likewise facilitate the application of the extended Euclidean geometry to many different types of practical problems arising in science, engineering, or industry. Often these applications require the use of a branch of mathematics which, like ordinary algebra, originates from the number system, but makes use of more elaborate techniques than algebra — namely, differential and integral calculus. This subject involves the study of functions by means of limiting processes. The methods of calculus are indispensable for the definition of the rather elusive geometric concepts of *lengths*, *areas*, and *volumes*, and for numerical evaluation of these in particular cases.

The methods of analytic geometry are also available for studying a broader kind of geometry than Euclidean, known as projective geometry. From the postulational point of view projective geometry is a generalization of both Euclidean and non-Euclidean geometry, in that no fundamental postulates or properties concerning such "metric" concepts as lengths and angles are involved, and there is complete duality between projective properties of points and lines in the plane.† By contrast with projective geometry, the Euclidean and non-Euclidean geometries are often referred to as *metric* geometries. Other

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* For a simplified discussion of angle trisection, see L. E. Dickson, "Why It Is Impossible to Trisect an Angle or to Construct a Regular Polygon of 7 or 9 Sides by Ruler and Compasses," *The Mathematics Teacher*, Vol. 14, 1921, pp. 217–223. See also Dickson, *New First Course in the Theory of Equations*, John Wiley, 1939. For more advanced discussions see the other Dickson reference in the Bibliography, or R. Courant and H. E. Robbins, *What is Mathematics?*, Oxford University Press, 1941, Chapter III.

† Typical figures studied in projective geometry are the *complete quadrilateral* and *complete quadrangle* (see Section 7–4). A typical theorem of projective geometry is the Desargues triangle theorem, which states that if the vertices of two triangles are in correspondence with each other in such a way that lines joining corresponding vertices are concurrent, then the points of intersection of corresponding sides are collinear.

classifications of geometries and subgeometries can be made from both the analytic and postulational points of view.

It should be noted that the process of studying geometric properties with the assistance of algebra and calculus can be reversed. Many problems involving algebraic operations or functional relations can be interpreted geometrically by the device of "graphing" pairs of values of two variables as points, and equations as curves. This is in line with the fact that a geometric interpretation is possible for the postulates of ordinary algebra. So if our primary concern is building up knowledge of algebra or function theory, we can translate problems into the geometric language as an aid to thinking, thus reversing the point of view of analytic geometry. This mutual reenforcement of methods of algebra or calculus on the one hand and geometry on the other is a striking feature of modern mathematics.*

It may be of interest to mention a somewhat informal set of postulates for Euclidean geometry which makes explicit use of the interrelation of algebra and geometry. These postulates, based on an idealized kind of measurement by scale and protractor, were proposed by the late American mathematician George D. Birkhoff in 1932, and were incorporated into an elementary textbook by Birkhoff and Beatley.† The postulates assume at the start a one-to-one correspondence between the points of a straight line and the real numbers, and thus are more closely identified than usual with the method of analytic geometry.

We now go on to the crucial question of the foundations of the algebraic number system.

11–5 Huntington's postulates for the algebra of complex numbers. A comprehensive postulational investigation of the real and complex number system of algebra was made near the beginning of the present century by E. V. Huntington. Some important results of

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* The men usually given credit for founding analytic geometry are the French mathematicians, Descartes (1596–1650) and Fermat (1601–1665). Newton (1642–1727) and Leibniz (1646–1716, German) share the credit for inventing calculus.

† G. D. Birkhoff, "A Set of Postulates for Plane Geometry, Based on Scale and Protractor," *Annals of Mathematics*, Vol. 33, 1932, pp. 329–345. G. D. Birkhoff and Ralph Beatley, *Basic Geometry*, Scott, Foresman and Co., 1940.

the investigation are given in a monograph of his which contains a set of 27 postulates for the algebra of complex numbers.* It is claimed that the set is consistent, independent, and categorical; and it is indicated how to modify the complete set so as to obtain categorical postulates for the real number system alone.

The 27 postulates are in abstract form, and the basic undefined terms consist of a class K of elements (corresponding to the complex numbers) and a subclass of K, designated as C (corresponding to the real numbers), together with two operations designated as $\oplus$ and $\odot$, and a relation $\circledleq$. (The circles are supposed to emphasize abstractness and to help avoid preconceived notions in testing the postulates.) The postulates can be summarized in consolidated form essentially as follows:

I. The elements of class K form a field with respect to $\oplus$ and $\odot$.

II. The subclass C is an ordered field.

III. The class C satisfies the following continuity postulate: Any nonempty subset S of C which has an upper bound in C has a least upper bound in C.

IV. There exists in K an element j whose square equals $\overline{u}$, where $\overline{u}$ is the opposite of the unity element of K (j is called the imaginary unit of K).

V. For every element a in K there exists a pair of elements x and y in C such that

$$x \oplus (j \odot y) = a.$$

Of these postulates, numbers I and II are familiar to us already. Numbers IV and V show the connection between the class of "complex" elements K and the subclass C of "real" elements.

Postulate III requires special mention. By an "upper bound in C," for a subset S, we mean an element b in C such that for every x belonging to S we have

$$x \leqq b \quad \text{(omitting the circle around the } \circledleq \text{ symbol).}$$

An element l is the "least upper bound" for S if there is no smaller

* E. V. Huntington, "The Fundamental Propositions of Algebra," in *Monographs on Topics of Modern Mathematics*, edited by J. W. A. Young, Longmans, Green and Co., 1927; also available as a separate publication (see the Bibliography).

upper bound, so that if $l' < l$ there is at least one element y of S such that

$$l' < y.$$

The significance of the postulate can be illustrated if we interpret S as the set of all rational numbers whose squares are less than 2, and C as the set of all real numbers. Then 1.5 is an upper bound for S, but it is not a *least* upper bound, since 1.42 is also an upper bound. The latter is not a least upper bound, since 1.415 is likewise an upper bound. Since no rational number when squared equals 2 but the square root of 2 can be approximated to any number of decimal places, we see that no rational number can be a least upper bound to S. Thus the continuity postulate would fail if C were interpreted as the system of rational numbers.

On the other hand, the required least upper bound does exist within C, using the specified interpretation of C as the system of real numbers. The real number $\sqrt{2}$ itself is the least upper bound. In this way, the real number system is considered free of gaps.

The main point for us to consider here is the question of consistency. Huntington uses a geometric interpretation of his undefined terms to justify his claim that the postulates are consistent. In this interpretation K is taken as the class of all points in the plane, and C as the class of all points on the "axis of reals," i.e., the x axis in analytic geometry. The operations $\oplus$ and $\odot$ are interpreted geometrically as is sometimes done in college algebra courses in connection with geometric representation of complex numbers and their operations.* $\leqslant$ is interpreted as an ordering relation of points on the axis of reals. In checking the postulates, use is made of familiar principles of Euclidean geometry, involving, for example, parallel lines and similar triangles. Accordingly the consistency of Huntington's postulates hinges on the consistency of Euclidean geometry.

This is unsatisfactory from our standpoint because we find ourselves in something of a vicious circle. Hilbert has contended that Euclidean geometry is consistent if the system of real (or complex) numbers is consistent, and now we find the contention reversed by Huntington. It appears then that we have no independent guarantee

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* For example, the sum of *points* a and b is taken to be the fourth vertex of a parallelogram whose other three vertices are the origin, point a, and point b, and whose sides are the vectors, or directed line segments, joining the origin to points a and b.

of the consistency of either Euclidean geometry or the number system of algebra; at best, each is consistent if the other is consistent.

Huntington did offer an alternative nongeometric type of consistency example, but this merely amounted to the concrete, definitional development of the number system. Thus, starting from the natural numbers

$$1, 2, 3, \ldots$$

he introduced successively, with appropriate definitions but without complete details, the systems of positive rationals, positive reals, all reals, and all complex numbers. The set of complex numbers thus constructed can be shown to satisfy all of the 27 postulates.

But does this latter fact in itself constitute an adequate consistency proof? In the process of proving that the various postulates are satisfied for the constructed system of complex numbers, the arguments must inevitably go back to the starting point of the definitional development, namely, the system of natural numbers. Accordingly, the best that can be claimed from this point of view is that the complex number system, or the set of 27 postulates proposed by Huntington for the complex number system, is consistent *if* the system of natural numbers is consistent. Huntington did not attempt to investigate the latter question, perhaps because previous work by Peano had already presented a postulational analysis of the natural numbers. We take up Peano's postulates in the next section.

Exercises

1. Investigate Huntington's proof of the categoricalness of his 27 postulates for the complex number system. (See his monograph "Fundamental Propositions of Algebra," cited in this section; pp. 191–192.)
2. Investigate some of Huntington's independence examples for his postulates. (See the monograph cited, pp. 192–195.)

11–6 Peano's postulates for natural numbers. There are two reasons why it is significant for us to include some discussion of the well-known postulates for natural numbers formulated by the Italian mathematician G. Peano (1858–1932).

First, as mentioned in the preceding section, a nongeometric consistency proof for a set of postulates for the complex number system, such as Huntington's, hinges on the consistency of the system of natural numbers. *Second*, we can indicate how to prove as theorems from Peano's postulates such fundamental properties of natural num-

bers as the commutative, associative, and distributive rules. Previously we have assumed these properties for all types of numbers, and have been content to say that these rules are established in the definitional development of the number system. We can use Peano's postulates to show how to get started in this definitional development from the level of the natural numbers.

PEANO'S POSTULATES FOR NATURAL NUMBERS *

Undefined terms: set of natural numbers, "successor of."

POSTULATE 1. 1 is a natural number.

POSTULATE 2. For every natural number x there exists a unique natural number which is the successor of x. (The successor of x is indicated by x'.)

POSTULATE 3. There is no natural number whose successor is 1.

POSTULATE 4. If two natural numbers x and y are such that their successors are equal, i.e., $x' = y'$, then $x = y$.

POSTULATE 5. *Principle of mathematical induction.* Let M be a subset of natural numbers with the following properties:

(a) 1 belongs to M,

(b) x belongs to M implies x' belongs to M.

Then M is the set of all natural numbers.

THEOREMS FOR NATURAL NUMBERS †

THEOREM 1. *For every natural number n, $n' \neq n$.*

Proof. Let M be the set of natural numbers x such that $x' \neq x$. By the first postulate, 1 is a natural number and by Postulate 3,

$$1' \neq 1.$$

Hence 1 belongs to M.

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* As presented by E. Landau, *Foundations of Analysis*, Chelsea Publishing Co., 1951.

For a slightly different version of the postulates see Bertrand Russell, *Introduction to Mathematical Philosophy*, second edition, Macmillan, 1924. See also C. G. Hempel, "On the Nature of Mathematical Truth," *American Mathematical Monthly*, Vol. 52, 1945, pp. 543–556.

For a somewhat more advanced discussion see R. B. Kershner and L. R. Wilcox, *The Anatomy of Mathematics*, Ronald Press, 1950, Chapter 8.

† Here we follow the general plan of development given by Landau, *op. cit.*, but certain details required for rigor are omitted.

Let y be a natural number belonging to M, so that

$$y' \neq y.$$

Then
$$(y')' \neq y'.$$

For if $(y')' = y'$, then we would have $y' = y$ by Postulate 4. Thus y' belongs to M.

Hence both conditions of the postulate of mathematical induction are satisfied, and M consists of all natural numbers. Thus for every natural number n we must have

$$n' \neq n.$$

Next we define addition of natural numbers "recursively" by the following definition in two parts.

DEFINITION 1. For any natural numbers x and y,

$$\left.\begin{aligned} x + 1 &= x' \\ x + y' &= (x + y)' \end{aligned}\right\}.$$

THEOREM 2. *If a is any natural number, then for all natural numbers y, $a + y$ is a natural number.*

Proof.

$a + 1 = a'$	by Definition 1.
a' is a natural number	by Postulate 2.
$\therefore a + 1$ is a natural number.	

Next, let x be a natural number such that $a + x$ is a natural number.

Then	$(a + x)' = a + x'$	by Definition 1,
and	$a + x'$ is a natural number	by Postulate 2.

Now form set M consisting of all natural numbers y for which $a + y$ is a natural number. By the first paragraph above,

1 belongs to M.

By the second paragraph,

x belongs to M implies x' belongs to M.

Hence by the postulate of mathematical induction M is the set of all natural numbers. That is, for all natural numbers y, $a + y$ is a natural number.

THEOREM 3. *If a and b are natural numbers, then*

$$(a + b) + 1 = a + (b + 1).$$

Proof. $a + b$ is a natural number, by Theorem 2.

$$\left.\begin{aligned}(a+b)+1 &= (a+b)' \\ &= a+b' \\ &= a+(b+1)\end{aligned}\right\} \text{by Definition 1.}$$

THEOREM 4. *If a, b are any natural numbers and if for a certain natural number x*

$$(a+b)+x = a+(b+x),$$

then $$(a+b)+x' = a+(b+x').$$

Proof. By Theorem 2, $a+b$ is a natural number. Also $(a+b)+x$ is a natural number, and

$$\begin{aligned}(a+b)+x &= a+(b+x) && \text{by hypothesis.}\\ \therefore [(a+b)+x]' &= [a+(b+x)]' && \text{by Postulate 2,}\end{aligned}$$

or

$$\begin{aligned}(a+b)+x' &= a+(b+x)' && \text{by Definition 1,}\\ &= a+(b+x') && \text{by Definition 1.}\end{aligned}$$

THEOREM 5. *If a, b are any natural numbers, then for all natural numbers y*

$$(a+b)+y = a+(b+y).$$

Proof. Let M be the set of all natural numbers y such that the desired associative law of addition holds. Then: (a) 1 belongs to M, by Theorem 3, (b) x belongs to M implies x' belongs to M, by Theorem 4. Hence, by the postulate of mathematical induction, M consists of all natural numbers.

THEOREM 6. *For all natural numbers y*

$$y+1 = 1+y.$$

Proof. First, $1+1 = 1+1$. Next, we let x be any natural number for which

$$x+1 = 1+x$$

and show that the same property holds for x'. This is done as follows:

$$\begin{aligned}x'+1 &= (x+1)+1 && \text{by Definition 1,}\\ &= (1+x)+1 && \text{by assumption,}\\ &= 1+(x+1) && \text{by Theorem 3 (or 5),}\\ &= 1+x' && \text{by Definition 1.}\end{aligned}$$

Let M be the set of natural numbers y for which the commutative property with respect to 1 holds. We have shown that the two induction conditions are satisfied, so M consists of all natural numbers.

THEOREM 7. *If a is any natural number, then for all natural numbers y*

$$a + y = y + a.$$

Proof. By Theorem 6,

$$a + 1 = 1 + a.$$

Now we assume that x is commutative with a and show that x' has the same property.

$$\begin{aligned} x' + a &= (x + 1) + a && \text{by Definition 1,} \\ &= (1 + x) + a && \text{by Theorem 6,} \\ &= 1 + (x + a) && \text{by Theorem 5,} \\ &= 1 + (a + x) && \text{by assumption,} \\ &= (a + x) + 1 && \text{by Theorem 6,} \\ &= a + (x + 1) && \text{by Theorem 5,} \\ &= a + x' && \text{by Definition 1.} \end{aligned}$$

The theorem now follows by the usual induction argument.

DEFINITION 2. For any natural numbers x and y

$$x \cdot 1 = x,$$
$$x(y') = xy + x.$$

THEOREM 8. *If a is any natural number, then for all natural numbers y, ay is a natural number.*

Proof. The proof is like that of Theorem 2, and is left as an exercise.

THEOREM 9. *If x, y are any natural numbers, then for all natural numbers z*

$$x(y + z) = xy + xz.$$

Proof. $x(y + 1) = xy + x \cdot 1$ by Definition 2. Thus the distributive law holds for $z = 1$. Now we suppose the law holds for w and show it holds for w'. That is, we assume

$$x(y + w) = xy + xw,$$

and get

$$\begin{aligned} x(y + w') &= x(y + w)' && \text{by Definition 1,} \\ &= x(y + w) + x && \text{by Definition 2,} \\ &= (xy + xw) + x && \text{by assumption,} \\ &= xy + (xw + x) && \text{by Theorem 5,} \\ &= xy + xw' && \text{by Definition 2.} \end{aligned}$$

Now let M be the set of all natural numbers z for which the distributive law holds in conjunction with x, y. Both induction conditions of Postulate 5 have been shown to hold. Hence M consists of all natural numbers.

By similar methods the following theorems can be proved. The proofs are left as exercises. (Theorem 10 is included here for use in proving Theorem 12.)

THEOREM 10. *If x and y are any natural numbers, then for all natural numbers z*

$$(x + y)z = xz + yz.$$

THEOREM 11. *For all natural numbers x, y, and z*

$$(xy)z = x(yz).$$

THEOREM 12. *For all natural numbers x, y*

$$x \cdot y = y \cdot x.$$

Thus it is possible to deduce from Peano's postulates and definitions the standard closure, commutative, associative, and distributive rules for natural numbers. This puts us in a position to continue with the definitional development of the number system, introducing the number 0, order relations, rational numbers, and so on through to the system of complex numbers.

Coming back to the consistency question, we can hardly claim that Peano's postulates help establish the consistency of the natural numbers. All we can say is that if we grant that the (concrete) system of natural numbers is consistent and that the natural numbers, as commonly used for counting and other concrete purposes, actually have the properties demanded by Peano's postulates, then these (abstract) postulates are consistent. There seems little doubt that the postulates correspond to our intuitive ideas about the natural numbers, but unfortunately our intuition sometimes deceives us.

Peano's postulates can be shown to be categorical on the assumption that they are consistent. This is not a great deal of help. We are still left with the crucial problem of the consistency of the concrete system of natural numbers.

The only way out is to push the starting level of the definitional development of the number system down still deeper, and attempt to derive the concrete properties expressed abstractly by Peano's postulates from a more secure foundation.

Mathematicians who attempt to do this usually are inclined to start with a systematic development of the theory of sets, relations, and one-to-one correspondence. Such concepts are actually involved in the abstract statement of Peano's postulates. By use of these and related ideas, definitions can be given both for the general concept of *cardinal number*, i.e., the *number* of elements in a class, and for operations with cardinal numbers. Theorems can then be deduced leading up to the desired properties of natural numbers from the *ordinal* point of view, as required by Peano's postulates. (The ordinal viewpoint simply looks at the natural numbers in the standard order used in counting.)

Logicians push the starting level of the development down still further and derive the theory of classes from a foundation in the logic of propositions and propositional functions.* We are well aware that Whitehead and Russell's *Principia Mathematica* is the most notable attempt of this kind to date. So we now turn our attention briefly to the foundation of logic presented there.

11–7 The Whitehead-Russell foundation for logic. Officially, the *Principia Mathematica* starts with "primitive ideas" and "primitive propositions" corresponding to what we have called "undefined terms" and "postulates." But there is this difference: the primitive ideas and propositions are not supposedly subject to interpretation in the usual postulational manner. They are specifically restricted to concepts of logic. In other words, a *concrete* rather than an *abstract* point of view prevails.

The chief aim of *Principia Mathematica* is to develop a usable body of logic leading up through the theory of classes and relations to the concrete foundations of the number system, and hence to all mathematics derivable from the number system. The school of thought which agrees with the corresponding philosophical view of Russell and Whitehead, that mathematics can thus be considered as an extension of logic, is sometimes known as *logicalism*.†

No attempt is made in *Principia Mathematica* to prove the con-

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* Chapter 4 suggested roughly how this might be started. We cannot hope to base this development satisfactorily on abstract postulates for Boolean algebra, inasmuch as we must assume some preliminary concrete knowledge of logic in order to deduce theorems from these postulates.

† For expositions of this point of view, see the references to Russell and Hempel in the first footnote of the preceding section.

sistency or independence of the primitive propositions. In fact, it is the opinion of Whitehead and Russell that the usual postulational methods are not suitable at such a fundamental level under the concrete point of view adopted.

The primitive propositions are used not merely as a substantive foundation for the deduction of a body of theorems, but also for furnishing the rules of procedure by which theorems may legitimately be deduced. Concurrent with the deduction process is the familiar process of combining the primitive concepts into new concepts through definitions. From the formal point of view, according to Whitehead and Russell, a definition is merely the substitution of a new symbol for a combination of symbols whose meaning is already known because they represent either primitive ideas or ideas previously defined. The authors hold that the defining process is theoretically superfluous; from the practical standpoint, however, it is surely indispensable and equal in importance to the deductive process.

Important among the primitive ideas are *assertion*, *negation*, and *disjunction* of propositions. Assertion of a proposition is in the sense of affirming a true statement. Negation and disjunction have the familiar meanings.

Implication is defined in terms of negation and disjunction, as we have indicated in Chapter 3:

$$p \to q \quad \text{means} \quad (\sim p) \vee q.$$

Conjunction, instead of being adopted as a primitive concept, is also defined in terms of disjunction and negation by the use of one of DeMorgan's principles. Thus

$$pq \text{ is defined as } \sim[(\sim p) \vee (\sim q)].$$

The primitive propositions include a rule of deduction which is essentially the same as our "fundamental rule of inference." The remaining primitive propositions are for the most part implications involving disjunctions. The *law of excluded middle* and the *law of contradiction* which we adopted as assumptions in Chapter 3 appear in *Principia Mathematica* as theorems.

The primitive propositions which appear in the form of implications are the following. (We list them by the numbers used to designate them in *Principia Mathematica*, but we have omitted the "assertion sign" indicating that these are asserted to be true, and we have changed the horseshoe implication symbol to our arrow.)

1.2. $(p \vee p) \rightarrow p$.
1.3. $q \rightarrow (p \vee q)$.
1.4. $(p \vee q) \rightarrow (q \vee p)$.
1.5. $[p \vee (q \vee r)] \rightarrow [q \vee (p \vee r)]$.
1.6. $(q \rightarrow r) \rightarrow [(p \vee q) \rightarrow (p \vee r)]$.

Some of the early theorems resulting from the primitive propositions are the following:

2.01. $(p \rightarrow \sim p) \rightarrow (\sim p)$. (Principle of *reductio ad absurdum*.)
2.02. $q \rightarrow (p \rightarrow q)$. (A true proposition is implied by any proposition.)
2.03. $(p \rightarrow \sim q) \rightarrow (q \rightarrow \sim p)$. (One form of the *opposite converse* principle.)
2.05. $(q \rightarrow r) \rightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)]$. (*Syllogism principle* analogous to our "chain rule.")

The proofs of these can be indicated briefly as follows, using in each case the definition of implication after substitution:

2.01. Substitute $\sim p$ in place of p in 1.2.
2.02. Substitute $\sim p$ for p in 1.3.
2.03. Substitute $\sim p$ for p and $\sim q$ for q in 1.4.
2.05. Substitute $\sim p$ for p in 1.6.

The above primitive propositions and theorems refer to the theory of deduction in terms of *elementary* propositions (corresponding to what we have called *particular* propositions). Additional primitive propositions are introduced in order to extend the theory of deduction to quantified propositions. This is followed by a lengthy and highly symbolic treatment of the theory of classes and relations, completing the first half of Volume I. The second half of the volume is devoted entirely to a "Prolegomena to Cardinal Arithmetic." The magnitude of the task can be somewhat appreciated by noting that this part arrives at definitions of the numbers 1 and 2, but the general definition of cardinal number is not reached until the beginning of Volume II. It must be granted that few mathematicians have the patience to follow through the *Principia's* introduction to elementary arithmetic!

As far as the consistency problem is concerned, there is little doubt that *Principia Mathematica* contains justification for Peano's postulates as properties of the system of natural numbers. Thus the consistency problem for Euclidean geometry, for the complex

number system, and for branches of mathematics depending on these, is shifted far back to the consistency of the foundations of the *Principia Mathematica*. As we have seen, nothing is done by Whitehead and Russell to establish this consistency, except to claim that the primitive propositions are true, and hence presumably consistent. But the claim is not supported except by informal considerations.

Furthermore, Whitehead and Russell make use of some special theories and axioms of a controversial nature. For example, the theory of *types* is an important subtheory introduced in *Principia Mathematica* to resolve certain bothersome paradoxes of logic. One of these paradoxes, due to Russell, is the following:

> Let C be defined as the class consisting precisely of *those classes which are not members of themselves*. Is C a member of itself? If it is a member of itself, then it must be one of the classes belonging to C and hence must be *not* a member of itself. On the other hand, if C is not a member of itself, it must be one of those classes which make up class C and so it *must be* a member of itself after all. In either case there is a contradiction.

According to the theory of types, it is improper and, in fact, meaningless to talk about such a class C. Similar paradoxes, which are supposedly resolved in *Principia Mathematica*, may involve propositions rather than classes. A very simple one is the following sentence:

This statement is false.

If false, it is true; and if true, it is false!

Although it is generally recognized by mathematical logicians that the *Principia Mathematica* has made a magnificent contribution to the foundations of mathematical knowledge, still this work is far from being accepted as perfect. Research in mathematical logic continues at a prolific rate at the present time, and there are plenty of controversial issues to stimulate continued research.

11–8 Formalist and intuitionist approaches to the foundations. Other systems of logic. One major controversy in the recent past has been between the *formalist* and *intuitionist* schools of thought in the foundations.

The formalist school, started by David Hilbert (after completing his postulational study of geometry), has had an aim somewhat similar to the Whitehead-Russell objective of carrying the foundation of

all mathematical knowledge down to a much more fundamental level than is customary. But whereas the *Principia Mathematica* was content to start with logic, Hilbert claimed that the ultimate foundation lies not in logic but in *prelogical* objects — namely, marks or symbols — which are preliminary conditions for logical thinking, and about which we seem to have intuitive knowledge.*

Hilbert and his followers used ideas of this kind to devise a formalized *theory of proof*, including within it a system of symbolic logic. The aim of the theory was to prove the consistency of the logical structure of ordinary mathematics. Inasmuch as the theory was outside of mathematics, it was termed *metamathematics*.

Any hope of success in the formalist goal of such a far-reaching consistency proof was apparently quashed by K. Gödel in his theory of consistency and completeness, dating from about 1930. Gödel showed essentially that within a formalized deductive system such as Hilbert's, it is impossible to prove consistency of the system by methods belonging to that system. Thus the consistency of this metamathematical system itself could not be guaranteed. This does not mean, however, that parts of Hilbert's theory may not turn out to be of lasting value.

Prior to Gödel, Hilbert and the formalist school came under attack by the Dutch mathematician, L. E. J. Brouwer, leader of the intuitionists. Brouwer asserted that Hilbert was making mathematics degenerate into a game. Hilbert was willing to admit that his proof theory was similar to a game, but claimed that it was a game with a serious purpose for all of mathematics.

The intuitionist point of view is opposed to the formalist emphasis on symbolism as the prerequisite to mathematical knowledge. The intuitionists believe that mathematical concepts have, or should have, psychological meaning in themselves, not through the medium of abstraction and symbolism. The concrete natural numbers and the principle of mathematical induction are key concepts in mathematical knowledge from the standpoint of the intuitionists. Furthermore, in order to establish the existence of numbers with a specified property, it is not considered legitimate proof from their standpoint

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* For a comparison of the formalist and logicalist schools and the somewhat vaguely defined *postulational* school, see E. R. Stabler, "An Interpretation and Comparison of Three Schools of Thought in the Foundations of Mathematics," *The Mathematics Teacher*, Vol. 28, 1935, pp. 5–35.

merely to reach a contradiction by assuming that for all numbers the property in question fails. *Explicit construction* of a number with the required property is demanded. This is understandable in view of the intuitionists' denial of the validity of the law of excluded middle for infinite sets.

In a sense, the intuitionists are extreme skeptics — in fact, so much so that they have been accused of casting doubt on the soundness of the foundations of modern *mathematical analysis*, a general term commonly used to designate calculus, differential equations, theory of functions, and related branches of mathematics.

It is probably fortunate that most mathematicians have not taken these doubts too seriously. But the intuitionists as well as the formalists have helped spur on critical research in the foundations, which will be quite likely to have beneficial results for mathematical rigor and mathematical philosophy.

For one thing, the intuitionist viewpoint has been an incentive to development of new systems of logic to conform with the intuitionist denial of the complete validity of the law of excluded middle. Such systems, of course, conflict both with classical logic and the Whitehead-Russell logic.

Still other systems of logic go beyond the intuitionist logic in novelty. For example, *many-valued logics* and *probability logics* have been developed in which a proposition may have various shades or degrees of truth-values — sometimes infinitely many — between falsity and truth at the two extremes. It is quite conceivable that such systems of logic may have useful practical applications. In some cases, these have been developed especially for inductive and empirical methods in scientific inquiry.*

On the more conservative side, a system of logic based on the concept of *strict implication* has been formulated.† The idea here was to replace the very broad definition of implication used in *Principia Mathematica* by a kind of implication which demands a more definite logical connection between two propositions.

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* For probability logic and inductive logic see, for example, H. Reichenbach, *The Theory of Probability*, University of California Press, 1949, Chapters 10 and 11.

† See C. I. Lewis and C. H. Langford, *Symbolic Logic*, Century Co., 1932; also E. V. Huntington, "Postulates for Assertion, Conjunction, Negation and Equality," *Proceedings of the American Academy of Arts and Sciences*, Vol. 72, 1937, pp. 1–44.

The discussion in this section has necessarily been sketchy because of the advanced nature of many of the topics. For a more adequate treatment we shall have to suggest supplementary reading; see, for example, the following additional references:

E. T. Bell, *The Development of Mathematics*, McGraw-Hill, 1940, pp. 522–536. (This discusses logicalism, intuitionism, and formalism, including the work of Gödel.)

Orrin Frink, "New Algebras of Logic," *American Mathematical Monthly*, Vol. 45, 1938, pp. 210–219.

Saunders MacLane, "Symbolic Logic," *American Mathematical Monthly*, Vol. 46, 1939, pp. 289–296. (This contains a bibliographical summary of many topics and points of view in the foundations of mathematics and logic.)

W. V. Quine, *Methods of Logic*, Henry Holt, 1950. (Section 41 discusses Gödel's theory.)

Hermann Weyl, "Mathematics and Logic," *American Mathematical Monthly*, Vol. 53, 1946, pp. 2–13; "A Half Century of Mathematics," *ibid.*, Vol. 58, 1951, pp. 523–553.

R. L. Wilder, "The Cultural Basis of Mathematics," *Proceedings of the International Congress of Mathematicians*, Vol. I, American Mathematical Society, 1952, pp. 258–271.

11–9 On mathematical knowledge from various foundational points of view. One thing seems certain as a result of our survey of mathematical foundations. No final proof of the consistency of ordinary mathematics (i.e., mathematics based on the real or complex number system or on Euclidean geometry) has yet been achieved. It seems doubtful whether any such final proof is ever to be expected.

Furthermore, no final agreement has been reached as to the "correct" system of logic to be used as the deductive instrument of mathematical proofs. Freedom of choice here seems somewhat similar to the freedom of choice available between Euclidean and non-Euclidean geometries. Thus there appears to be no necessity to think of the Whitehead-Russell type of approach to the concrete foundations of the number system as the only possible road to take. Some other approach may turn out to be preferable.

Perhaps at the present stage of mathematical development we have to be content with the broad pragmatic point of view — that ordinary mathematics has achieved worth-while results from both the pure and applied standpoints. Surely great progress has been

made since the time of Euclid in two main directions: in building up powerful new systems of mathematics adapted to many different purposes, and in delving down to make logical methods more rigorous though more flexible, and thus making the foundations of mathematics more secure.

Whether we adopt the viewpoint of the *postulationists* (as represented for example by Huntington and the geometric foundational work of Hilbert), the *logicalists* (as represented by the *Principia Mathematica* of Whitehead and Russell), the *formalists* (as represented by the proof theory and metamathematics of Hilbert and his followers), the *intuitionists* (as represented by Brouwer), or some other point of view in the foundations, it seems clear that the whole structure of mathematics rests ultimately on unproved principles and on intuitively given concepts at some level. The best hopes for the security of mathematical knowledge seem to depend on the psychological integrity of the human mind, supplemented by the human capacity to improve ideas transmitted from one generation to the next.

11–10 Retrospect. This text has avowedly been introductory and exploratory. Doubtless it has been guilty of redundancies in several places, but it has seemed desirable to emphasize certain ideas repeatedly from a variety of points of view. (It is hoped that there has been no similar guilt from the standpoint of consistency.)

A number of topics have been ignored or slighted which may well have deserved considerable discussion. Among these are various topics important for a more advanced understanding of the foundations of mathematics — for example, the theory of infinite sets, well-ordered systems, axiom of choice, transfinite numbers, and basic concepts of topology.

On the philosophical side it might have been profitable to consider the relations of mathematics and language (including semantics), and of mathematics and art. Certainly a well-developed logical structure of mathematical ideas should have some of the esthetic characteristics of a work of art; and certainly the development of ideas in modern logic and mathematics calls for a special medium, a symbolism, in addition to the symbolism of ordinary language. Thus some characteristics common to mathematics and music should be immediately apparent. Music has been called both an art and a language, and "logical development" of musical ideas is often mentioned by music

critics. A useful comparison might be made between mathematical thought and musical thought, as represented by good examples of mathematical systems and musical compositions. We leave this, however, as an optional exercise for musically inclined readers!

For those readers who desire a recapitulation of both parts of the text, we need only mention several outstanding themes: (1) the importance of postulational organization and methods in both pure and applied mathematics, and their significance for science and many other fields of thought; (2) the existence of key patterns of logical reasoning as used in logical systems and the urgent need of a critical attitude toward reasoning in numerous situations of common occurrence; (3) the relative rather than absolute nature of the truth of theorems in mathematics or of statements purportedly deduced from assumptions in any domain of thought; (4) the lack of any absolute guarantee of consistency in ordinary mathematics and the ultimate dependence of mathematical security on intuitive concepts and the sanity of the human mind; (5) the determination of usefulness of mathematical problems, concepts, and theories, by the extent to which these contribute to fruitful or unifying developments within mathematics or in other areas, regardless of whether the subject matter concerned is classified as pure or applied; (6) the evolving unfinished nature of mathematical, logical, and scientific knowledge.

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